AIChE

Distributed Model Predictive Control Based on Dissipativity

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DOI 10.1002/aic.13868
Published online June 27, 2012 in Wiley Online Library (wileyonlinelibrary.com).

A noncooperative approach to plant-wide distributed model predictive control based on dissipativity conditions is developed. The plant-wide process and distributed control system are represented as two interacting process and controller networks, with interaction effects captured by the dissipativity properties of subsystems and network topologies. The plant-wide stability and performance conditions are developed based on global dissipativity conditions, which in turn are translated into the dissipative trajectory conditions that each local model predictive control MPC must satisfy. This approach is enabled by the use of dynamic supply rates in quadratic difference forms, which capture detailed dynamic system information. A case study is presented to illustrate the results. © 2012 American Institute of Chemical Engineers *AIChE J,* 59: 787–804, 2013

Keywords: process control, distributed model predictive control, dissipativity, dynamic supply rate, quadratic difference forms

Introduction

Model predictive control (MPC) is perhaps the most successful modern control approach applied in the chemical industry, with a large, and growing, number of installations globally.[1] MPC can explicitly handle operational constraints and has the ability to generate an optimal control sequence, which are of particular importance in chemical process control practice. Along with constraints, the large scale of plant-wide control problems and strong interactions between units present significant challenges in control practice. Mass recycle and heat integration are common in modern chemical plants, which are often designed to optimize steady state efficiency and operation, not dynamic operability. These recycle streams, which may be implemented at both the unit and plant-wide level, represent positive feedback loops, which can be deleterious to control performance. The complexity of modern chemical plants means that a centralized plant-wide control approach is often not practical. The interactions, scale, and complexity of modern chemical plants and the computational load required for plant-wide MPC suggests a distributed approach to plant-wide control is appropriate, for example, in Dunbar[2] the distributed MPC is shown to yield superior performance than decentralized MPC.

In distributed MPC strategies local controllers are allowed to communicate with one another to improve predictions and global performance while distributing the large computational load. However, the coordination of the individual controllers is still an open problem. It has been shown that modeling interactions and exchanging trajectories alone are not sufficient to ensure plant-wide overall stability.[3] The typical strategies for ensuring MPC stability based on terminal constraints and terminal costs (e.g., Maciejowski[4] and Mayne et al.[5]) cannot be used to ensure plant-wide stability as the effects of interconnections and interactions between subsystems are not taken into account. As such, it is necessary to implement plant-wide stability conditions for distributed MPC.

Most recent developments are focused on cooperative distributed MPC approaches, where the controllers exchange trajectory information to solve a global optimization problem in their local input variables. This is in contrast to noncooperative MPC, where the controllers optimize a local cost function. It has been found that the Pareto optimal solution can be obtained if controllers are allowed to iterate to convergence and the closed-loop stability is assured even when the iteration of the distributed controllers is terminated before convergence.[6] This is extended by introducing a method to decrease communication overheads.[7] In Jia and Krogh,[8] a stability constraint is placed on the state at the next time step so as to allow short prediction horizons to be employed. Approaches have been developed to implement a distributed MPC system to coordinate with a pre-existing lower level control system.[9,10] In Liu et al.[11] iterative and sequential nonlinear distributed MPC strategies are provided in the case of communication delays and asynchronous sampling. A detailed description of such distributed MPC given in Christofides et al.[12]

In this article, based on the dissipativity theory, a noncooperative plant-wide distributed MPC approach is developed. The plant-wide process and distributed control system are represented as two interacting process and controller networks, with interaction effects captured by the dissipativity properties of subsystems and network topologies. The plant-wide stability and performance conditions are developed based on global dissipativity conditions, which in turn, are

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translated into the dissipativity conditions that each local MPC must satisfy. This dissipativity assuring constraint is then added to the online MPC algorithm. In this way, each controller carries out individual (local) optimization based on its local sensor output and the information from other controllers, while the stability and minimum performance conditions are formulated at a global level. Compared to cooperative approaches, the proposed control is simpler to implement and may require less communication, yielding a more scalable approach suitable for distributed autonomous controllers. It also allows for arbitrary control structure, from fully decentralized control to distributed control with arbitrary controller communication network topologies.

In this work, the dissipativity conditions are used to capture the interactions effects and ensure plant-wide stability and performance. A useful feature of dissipativity based analysis is that it allows for complex interconnections between systems to be formulated in a "linear" manner, as the storage function and supply rate of the overall interconnected system, are linear combinations of the storage functions and supply rates of individual systems, respectively (based on the network topology). This makes dissipativity theory an ideal tool for analysis and control of large-scale systems. Focused on the interaction effects, the proposed approach is different to the existing dissipativity formulation in MPC developments where the dissipativity conditions are used to ensure the stability of single closed-loop systems. For example, the work of Løvaas and coworkers[13,14] ensures the stability of robust MPC policies for single systems. In Raff et al.[15] the passivity theorem is used to ensure the closed-loop stability for passive processes. In Chen and Scherer[16] a dissipativity ensuring constraint is placed on the online MPC algorithm, which allows for guaranteed $\mathcal{H}_\infty$ performance. Distributed $\mathcal{H}_\infty$ control has also been explored in Rice and Verhaegen.[17]

The traditional concept of dissipativity is defined as a property of the input and output spaces of a system which does not apply to MPCs without a closed form. Therefore, the concept of the dissipative trajectories is introduced in this article. Another key feature of this work is the use of dynamic supply rates and storage functions as quadratic difference forms (QDFs). QDFs include not only information on the current input and output of systems, but also on the predicted future inputs and outputs, thus, capturing much more detailed information on the process dynamics than the traditional static supply rates. This leads to less conservative stability results, as shown in our earlier work,[18] and more detailed global performance designs.

The remainder of the article is structured as follows: The next section provides a review of the relevant concepts of dissipativity and QDFs, and introduces dissipative trajectories. Then, a network view of plant-wide processes and distributed control is presented, followed by the plant-wide dissipativity analysis. Next, the global stability and minimum performance bounds are formulated in a unified way in terms of dissipativity conditions. This is followed by the proposed distributed control design approach. Finally, an illustrative example is presented to demonstrate the application of the proposed control approach.

Dissipativity and Dynamic Supply Rates

First introduced in Willems,[19] dissipative systems are those for which the increase in stored energy is bounded by the amount of energy supplied by the environment. As an input-output property of systems, dissipativity is useful in studying interconnected systems as it allows for much of the complexity of the problem to be shifted to the interconnection relations, rather than studying centralized models. Once the dissipativity of the subsystems is ascertained, the dissipativity based analysis for complex networks can be performed easily, yielding a scalable approach. A discrete time dynamical system with input, output and state u, y, and x, respectively, is said to be dissipative if there exists a function defined on the input and output variables, called the supply rate $s(u,y)$ and positive semidefinite function defined on the state, called the storage function $V(x(t))$ such that[19]

$$V(x(t+1)) - V(x(t)) \le s(u(t), y(t)) \tag{1}$$

for all time steps t. The following (Q,S,R)-type of supply rate is commonly used

$$s(u(t), y(t)) = y^T(t)Qy(t) + 2y^T(t)Su(t) + u^T(t)Ru(t) \tag{2}$$

Quadratic differential forms were first introduced in Willems and Trentelman[20] in the context of continuous time behavioral systems. This framework was then extended to the discrete time case in Kojima and Takaba.[21,22] A quadratic difference form (QDF) may be written in terms of extended inputs and outputs. Defining

$$\begin{aligned} \hat{u}^T(t) &= \begin{pmatrix} u^T(t) & u^T(t+1) & \dots & u^T(t+\tilde{n}) \end{pmatrix} \\ \hat{y}^T(t) &= \begin{pmatrix} y^T(t) & y^T(t+1) & \dots & y^T(t+\tilde{m}) \end{pmatrix} \end{aligned} \tag{3}$$

(for some finite $\tilde{n}$ and $\tilde{m}$), a QDF supply rate, denoted Q_ϕ, is defined as follows

$$Q_\phi(y,u) = \hat{y}^T(t)\tilde{Q}\hat{y}(t) + 2\hat{y}^T(t)\tilde{S}\hat{u}(t) + \hat{u}^T(t)\tilde{R}\hat{u}(t) \tag{4}$$

$$Q_\phi(y,u) = \begin{pmatrix} \hat{y}(t) \\ \hat{u}(t) \end{pmatrix}^T \begin{pmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{pmatrix} \begin{pmatrix} \hat{y}(t) \\ \hat{u}(t) \end{pmatrix} \tag{5}$$

Essentially, a QDF is a quadratic form similar to (2), although it is extended to include future inputs and outputs. This allows for a more detailed description of the system in terms of dissipativity, and fits in well with the MPC framework, which is heavily based around prediction. This additional system information allows for less conservative stability and performance results as compared to other dissipativity based approaches. Defining $w(t) = [\hat{y}(t)^T, \hat{u}(t)^T]^T$, a QDF in (5) can be written in a compact form as

$$Q_\phi(y,u) = \sum_{k=0}^{N}\sum_{l=0}^{N} w^T(t+k)\phi_{kl}w(t+l) \tag{6}$$

where N is called the degree of supply rate, which is the maximum number of forward steps in the supply rate. In the supply rate given in (3)–(5), $N = \max\{\tilde{n}, \tilde{m}\}$. Such a QDF is said to be induced by the symmetric two variable polynomial matrix $\phi(\zeta, \eta)$ defined as

$$\phi(\zeta,\eta) = \sum_{k=0}^{N}\sum_{l=0}^{N} \zeta^k \phi_{kl}\eta^l \tag{7}$$

Here ϕ_{kl} is the matrix associated with the $w^T(t+k)\phi_{kl}w(t+l)$ term in (6). The indeterminates ζ and η represent a forward step in time on the left and right of $\phi(\zeta, \eta)$ respectively, that is, the forward steps in time of $w^T(t)$ and $w(t)$ respectively. The coefficient matrix of $\phi(\zeta, \eta)$, denoted $\tilde{\phi}$, is a (constant) matrix with (*k*,*l*)th block being ϕ_{kl}. That is

$$\tilde{\phi} = \begin{pmatrix} \phi_{00} & \dots & \phi_{0l} \\ \vdots & \ddots & \vdots \\ \phi_{k0} & \dots & \phi_{kl} \end{pmatrix} \tag{8}$$

Note that by permuting $\tilde{\phi}$ we can obtain the matrix in Eq. 5.

In this work, QDFs are used for both supply rates and storage functions. In this case (as in the behavioral systems theory[20]), a storage function is defined on the extended input and output (which is different to the conventional definition where the storage function is function of the state variables). The dissipativity condition with a supply rate Q_ϕ and a storage function Q_ψ in the QDF can be represented as follows

$$\sum_{t=0}^{\infty} Q_\phi(w(t)) \geq Q_\psi(w(t)) \tag{9}$$

A useful property of QDFs is that the rate of change of a QDF is itself a QDF, *i.e.*, $\nabla Q_\phi = Q_{\nabla\phi}$. That is, the rate of change of the QDF induced by $\phi(\zeta, \eta)$ is itself a QDF induced by $\nabla\phi(\zeta, \eta)$. This simplifies the calculation of the rate of change of a QDF as $\nabla\phi(\zeta, \eta) = (\zeta\,\eta - 1)\,\phi(\zeta, \eta)$. Equation 1 may then be written as

$$Q_{\nabla\psi}(w(t)) \leq Q_\phi(w(t)) \tag{10}$$

If $Q_\phi(w(t)) > 0\ \forall w(t) \neq 0$ then $\phi(\zeta, \eta)$ is said to be positive definite, with $\phi(\zeta, \eta) > 0$. Note that $\tilde{\phi} > 0 \Rightarrow \phi(\zeta, \eta) > 0$. Some results underlying dissipativity and its links to stability in this framework are briefly discussed below.

Theorem 1 (Kojima and Takaba[21]). *A discrete time linear time-invariant (LTI) system is asymptotically stable if there exists a symmetric two variable polynomial matrix* $\psi(\zeta, \eta) \geq 0$ *and* $\nabla\psi < 0$ *for all input and output satisfying the system equations*.

Note that the asymptotic stability defined above is for the outputs not states of a system. To ensure asymptotic stability of the states the additional (mild) assumption of zero state detectability is required.

In this article, state space models will be used in the prediction of future process outputs due to their efficiency in MPC implementation as compared to input–output models. The following result gives a linear matrix inequality (LMI) condition for determining the QDF dissipativity of a system with a state space representation. It is different to the existing result in Belur and Trentelman,[23] which is based on a different (kernel) system representation and for continuous time systems. One of the advantages of Proposition 1 is that it has fewer decision variables compared to the method in.[23]

Proposition 1. *A discrete time LTI system with state space representation* (*A*, *B*, *C*, *D*) *is dissipative with the supply rate and (positive semidefinite) storage function pair induced by* $\phi(\zeta, \eta)$ *and* $\psi(\zeta, \eta)$ *respectively, with the corresponding coefficient matrices* $\tilde{\phi}$ *and* $\tilde{\psi}$ *partitioned as* $\tilde{\phi} = \begin{pmatrix} \tilde{\phi}_Q & \tilde{\phi}_S \\ \tilde{\phi}_S^T & \tilde{\phi}_R \end{pmatrix}$ *and* $\tilde{\psi} = \begin{pmatrix} \tilde{\psi}_X & \tilde{\psi}_Y \\ \tilde{\psi}_Y^T & \tilde{\psi}_Z \end{pmatrix}$, *if and only if the following LMIs are satisfied*

$$\begin{pmatrix} \mathbb{T}_{11} & \mathbb{T}_{12} \\ \mathbb{T}_{12}^T & \mathbb{T}_{22} \end{pmatrix} \geq 0 \tag{11}$$

$$\tilde{\psi} \geq 0, \tag{12}$$

with

$$\hat{C} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^N \end{pmatrix} \tag{13}$$

$$\hat{D} = \begin{pmatrix} D & 0 & \dots & 0 & 0 \\ CB & D & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ CA^{N-1}B & CA^{N-2}B & \dots & CB & D \end{pmatrix} \tag{14}$$

$$\begin{aligned} \mathbb{T}_{11} &= \hat{C}^T[\tilde{\phi}_Q - \tilde{\nu}_X]\hat{C} \\ \mathbb{T}_{12} &= \hat{C}^T[\tilde{\phi}_Q - \tilde{\nu}_X]\hat{D} + \hat{C}^T[\tilde{\phi}_S - \tilde{\nu}_Y] \\ \mathbb{T}_{22} &= \hat{D}^T[\tilde{\phi}_Q - \tilde{\nu}_X]\hat{D} + \hat{D}^T[\tilde{\phi}_S - \tilde{\nu}_Y] \\ &\quad + [\tilde{\phi}_S - \tilde{\nu}_Y]^T\hat{D} + [\tilde{\phi}_R - \tilde{\nu}_Z] \end{aligned}$$

where N is the degree of the supply rate and $\nu(\zeta, \eta) = \nabla\,\psi(\zeta, \eta)$.

Proof. $\tilde{\psi} \geq 0$ implies that $\psi(\zeta, \eta) \geq 0$, and thus that the storage function $Q_\psi(u,y) \geq 0$. By taking forward steps in time

$$\begin{pmatrix} y(t) \\ y(t+1) \\ \vdots \\ y(t+N) \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^N \end{pmatrix} x(t) + \begin{pmatrix} D & 0 & \dots & 0 & 0 \\ CB & D & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ CA^{N-1}B & CA^{N-2}B & \dots & CB & D \end{pmatrix} \begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+N) \end{pmatrix} \tag{15}$$

Using the notation defined in (3) and with $N = \tilde{n} = \tilde{m}$, we have

$$\hat{y}(t) = \hat{C}x(t) + \hat{D}\hat{u}(t) \tag{16}$$

The dissipation inequality

$$Q_\phi(u(t), y(t)) \geq Q_{\nabla\psi}(u(t), y(t)) \tag{17}$$

can then be pushed to the extended input-output space and rearranged to obtain

$$\begin{pmatrix} x \\ \hat{u} \end{pmatrix}^T \begin{pmatrix} \mathbb{T}_{11} & \mathbb{T}_{12} \\ \mathbb{T}_{12}^T & \mathbb{T}_{22} \end{pmatrix} \begin{pmatrix} x \\ \hat{u} \end{pmatrix} \geq 0$$

from which the result in (11) is obvious.

In this work, the dissipativity of the process units is represented based on the above definition, which describes the relation between the input and output spaces of a system. Given its model, the dissipativity of a process unit can be determined by solving the LMI in Proposition 1, with decision variables $\tilde{\phi}_Q$, $\tilde{\phi}_S$, $\tilde{\phi}_R$, $\tilde{\nu}_X$, $\tilde{\nu}_Y$, and $\tilde{\nu}_Z$.

The proposed dissipativity based design is essentially on the basis of the plant-wide supply rates, rather than the storage functions. This allows the plant-wide stability and performance conditions to be directly related to the interactions

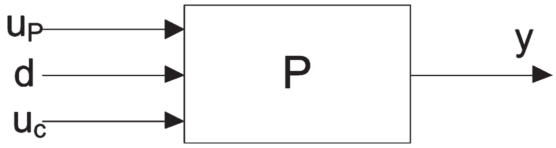

Figure 1. Partitioning of process manifest variables.

between process units, which are captured by the supply rates as functions of the interconnecting inputs and outputs. This approach also provides a unified framework that achieves both plant-wide stability and performance by optimizing the supply rates of process units and controllers. This is in contrast to many existing approaches discussed in section where the Lyapunov/storage functions are the focus of analysis.[3,11]

The supply rate induced by $\phi(\zeta, \eta)$ is equivalent to

$$s(u(t), y(t)) = \hat{y}^T(t)\tilde{\phi}_Q\hat{y}(t) + 2\hat{y}^T(t)\tilde{\phi}_S\hat{u}(t) + \hat{u}^T(t)\tilde{\phi}_R\hat{u}(t) \tag{18}$$

In a conceptually different way, the dissipativity of the MPC controllers is defined as a constraint on its input–output trajectory as defined below. ■

Definition 1 Dissipative Trajectory). *A MPC is said to trace a dissipative trajectory if, at all time instances k, the following dissipative trajectory inequality is satisfied*

$$W_k = \sum_{t=0}^{k} Q_{\phi_c}(u(t), y(t)) \geq 0 \tag{19}$$

where Q_{ϕ_c} is the supply rate induced by

$$\phi_c(\zeta, \eta) = \begin{pmatrix} Q_c(\zeta, \eta) & S_c(\zeta, \eta) \\ S_c^T(\zeta, \eta) & R_c(\zeta, \eta) \end{pmatrix} \tag{20}$$

The dissipative trajectory inequality ensures that the accumulated supply of the controller is nonnegative. This is in contrast to the usual discrete time dissipativity condition given in (9). The concept of a dissipative trajectory may be viewed as a slightly weaker version of the classic dissipativity as it does not explicitly require the dissipative trajectory inequality to be satisfied for all possible input $u(t)$. The MPC does not have a storage function, only a supply rate and an accumulation of supply. As such, if the controller has been "sufficiently dissipative" in the past, i.e., more dissipation has occurred earlier in the trajectory than the minimum amount, then this accumulation of supply may be used as a storage to compensate for deficiencies in dissipation in the future.

Plant-Wide Dissipativity Formulation

0.1 process and controller networks

The plant-wide chemical process is decomposed into a network of process units (subsystems) interacting through physical (mass and energy) flows. The dissipativity of the plant-wide system is determined based on the dissipativity of each subsystem and their interconnections. In this article, only unidirectional flows with time invariant network topology are considered, which are typical in plant-wide chemical processes where flows between units are usually forced by pumps or compressors. Similarly, the distributed control system is represented as a network of controllers communicating through an information network. In this work, the topologies of the process and controller networks are allowed to be arbitrary, represented by constant interconnecting matrices.

Consider a plant-wide chemical process with n process units. Partitioning the input to each process unit into the input from interconnected processes u_p, external disturbances d and manipulated input u_c, the i-th process unit P_i can be represented as follows

$$P_i : \left\{ \begin{pmatrix} x_i(k+1) \\ y_i(k) \end{pmatrix} = \begin{pmatrix} A_i & B_{1_i} & B_{2_i} & B_{3_i} \\ C_i & D_{1_i} & D_{2_i} & D_{3_i} \end{pmatrix} \begin{pmatrix} x_i(k) \\ u_{c_i}(k) \\ u_{p_i}(k) \\ d_i(k) \end{pmatrix} \right. \tag{21}$$

as shown in Figure 1. Thus, if a QDF supply rate of the i-th process is $Q_{\phi_i}(u_p, d, u_c, y)$, it is induced by a symmetric two variable polynomial matrix which may be conformally partitioned as

$$\phi_i(\zeta, \eta) = \begin{pmatrix} \mathcal{Q}_i(\zeta, \eta) & \mathcal{S}_i(\zeta, \eta) \\ \mathcal{S}_i^T(\zeta, \eta) & \mathcal{R}_i(\zeta, \eta) \end{pmatrix} \tag{22}$$

with

$$\mathcal{Q}_i(\zeta, \eta) = Q(\zeta, \eta)$$
$$\mathcal{S}_i(\zeta, \eta) = \begin{pmatrix} S_I(\zeta, \eta) & S_d(\zeta, \eta) & S_L(\zeta, \eta) \end{pmatrix}$$
$$\mathcal{R}_i(\zeta, \eta) = \begin{pmatrix} R_{II}(\zeta, \eta) & R_{Id}(\zeta, \eta) & R_{IL}(\zeta, \eta) \\ R_{Id}^T(\zeta, \eta) & R_{dd}(\zeta, \eta) & R_{dL}(\zeta, \eta) \\ R_{IL}^T(\zeta, \eta) & R_{dL}^T(\zeta, \eta) & R_{LL}(\zeta, \eta) \end{pmatrix}$$

As shown in the upper part of Figure 2, process units are stacked diagonally to form the overall plant without interconnections, denoted as $\tilde{\mathcal{P}}$. The inputs and outputs of this system are the vectors consisting of the inputs and outputs of each system. For example, the output of this block diagonal system is $\mathbf{y} = (y_1^T \dots y_n^T)^T$. The inputs $\mathbf{u}_p$, $\mathbf{d}$, and $\mathbf{u}_c$ are defined in the same way. The supply rate of this system is described as a QDF induced by the matrix

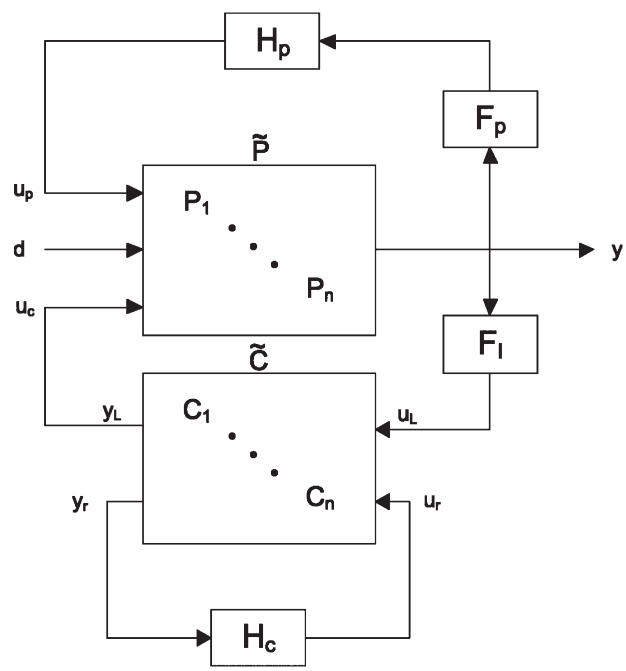

Figure 2. Network view of large-scale systems with distributed controller.

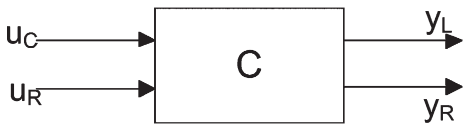

Figure 3. Partitioning of controller variables.

$$\Phi(\zeta,\eta)=\begin{pmatrix}\mathcal{Q} & \mathcal{S}\\ \mathcal{S}^T & \mathcal{R}\end{pmatrix} \tag{23}$$

with $\mathcal{Q}=\mathrm{diag}(\mathcal{Q}_1,\ldots,\mathcal{Q}_i,\ldots,\mathcal{Q}_n)$ and $\mathcal{S}$ and $\mathcal{R}$ similarly defined. Knowing each $\phi_i(\zeta,\eta)$ $(\theta_i(\zeta,\eta))$, $i=1,\ldots,n$, $\Phi(\zeta,\eta)$ $(\Theta(\zeta,\eta))$ can be be easily calculated. The relation between the storage function of the i-th process, induced by $\psi_i(\zeta,\eta)$, and storage function of $\tilde{\mathcal{P}}$, induced by $\Psi(\zeta,\eta)$, is completely analogous. The ability to easily formulate overall system dynamic features by considering only the features of the individual subsystems and interconnection relations is one feature of dissipativity based analysis which highlights its suitability in this context.

The distributed controller network is represented in a similar way. As shown in Figure 3, each controller has two pairs of input and output: one set of local input (local sensor output) u_c and output (manipulated variable) y_L; one set of remote input (information received from other controllers) u_r and output (information sent to other controllers) y_r. Assume that the supply rate of the i-th controller is induced by $\theta_i(\zeta,\eta) = \begin{pmatrix}\mathcal{Q}_{ci} & \mathcal{S}_{ci}\\ \mathcal{S}_{ci}^T & \mathcal{R}_{ci}\end{pmatrix}$. Then $\mathcal{Q}_{ci}$ may be partitioned as $\begin{pmatrix}\mathcal{Q}_{ci}^{LL} & \mathcal{Q}_{ci}^{Lr}\\ \mathcal{Q}_{ci}^{Lr^T} & \mathcal{Q}_{ci}^{rr}\end{pmatrix}$ with $\mathcal{S}_{ci}$ and $\mathcal{R}_{ci}$ being analogously partitioned. The supply rate of the system composing of the diagonal stacking of all controllers, $\tilde{\mathcal{C}}$, is then induced by $\Theta(\zeta,\eta)=\begin{pmatrix}\mathcal{Q}_c & \mathcal{S}_c\\ \mathcal{S}_c^T & \mathcal{R}_c\end{pmatrix}$ analogously as for the process units.

The overall process and controller networks are depicted in Figure 2. Matrices F_p and F_I represent "filters" which select the interconnecting and measured outputs respectively, thus are considered in this work constant with elements either 0 or 1. F_I can be considered as dynamical systems representing sensor dynamics. Matrix H_p represents the process network topology. It is constant with static interconnections. The interconnection relations may be described as $u_p=H_pF_py$. Similarly, H_c defines the controller network topology, which may have any structure, representing an arbitrary controller network topology.

Remark 1. *A common choice is the controller network to have the same topology as the process network (i.e., $H_c=H_p$), which ensures plant-wide stabilizability.*[24] *Another case is fully decentralized control, where the MPCs do not communicate with each other, which is represented by $H_c=0$.*

For example, consider the case study presented in Figure 4. The outputs of the reactors are the mass fractions of A and B (x_{A_i} and x_{B_i}) and the outlet temperature (T_i), while the outputs of the separator are the mass fractions of A and B in the tops (x_{A_T} and x_{B_T}), bottoms (x_{A_B} and x_{B_B}) and temperature of the top stream T_T. As all of these outputs except the bottoms mass fractions are part of the process network F_p takes the following form

$$F_p=\begin{pmatrix}I_{9\times9} & 0_{9\times2}\end{pmatrix} \tag{24}$$

so that the bottom two elements of $\mathbf{y}$, x_{A_B}, and x_{B_B} are "filtered out." The topology of the process network can be represented as

$$H_p=\begin{pmatrix}0_{3\times3} & 0_{3\times3} & I_{3\times3}\\ I_{3\times3} & 0_{3\times3} & 0_{3\times3}\\ 0_{3\times3} & I_{3\times3} & 0_{3\times3}\end{pmatrix} \tag{25}$$

showing that the input to the first process is the output of the third, the input of the second is the output of the first, and so on. Assuming that the controller network topology mimics the process network topology, H_c has the same form as H_p. This is a common, and intuitive, assumption that is used in literature both in the context of MPC and more general controller forms.[25]

Plant-wide dissipativity

Using the formulation in Figure 2, it is possible to determine the supply rate of the overall system with external input $\mathbf{d}$ and output $\mathbf{y_{pw}}=(y^T\ y_L^T)^T$, which is described in Lemma 1. Note that as the distributed controller traces a dissipative trajectory, so does the plant-wide system. However, the plant-wide system traces a dissipative trajectory for all external disturbances.

Lemma 1. *For the interconnected system as shown in Figure 2 with n processes and controllers, where $\tilde{\mathcal{P}}$ is dissipative with respect to supply rate Q_Φ and storage function $Q_\Psi\geq 0$, $\tilde{\mathcal{C}}$ traces a dissipative trajectory with respect to the supply rate Q_Θ and the process and controller networks have topology given by H_p and H_c respectively. For all external disturbances and all $k\geq 0$ the plant-wide system satisfies*

$$\sum_{i=0}^{k} Q_\mu(\mathbf{w}(i))\geq Q_\Psi(\mathbf{w}(i))\geq 0 \tag{26}$$

That is, it traces a dissipative trajectory with supply rate $Q_\mu(\mathbf{w})$ with

$$\mu(\zeta,\eta)=\begin{pmatrix}\Upsilon_{11}(\zeta,\eta) & \Upsilon_{12}(\zeta,\eta) & \Upsilon_{13}(\zeta,\eta)\\ \Upsilon_{12}^T(\zeta,\eta) & \Upsilon_{22}(\zeta,\eta) & \Upsilon_{23}(\zeta,\eta)\\ \Upsilon_{13}^T(\zeta,\eta)) & \Upsilon_{23}^T(\zeta,\eta) & \Upsilon_{33}(\zeta,\eta)\end{pmatrix} \tag{27}$$

where,

$$\Upsilon_{11}(\zeta,\eta)=\begin{pmatrix}\mathbb{X}_{11} & \mathbb{X}_{12} & \mathbb{X}_{13} & \mathbb{X}_{14}\\ \mathbb{X}_{12}^T & \mathbb{X}_{22} & \mathbb{X}_{23} & \mathbb{X}_{24}\\ \mathbb{X}_{13}^T & \mathbb{X}_{23}^T & \mathbb{X}_{33} & \mathbb{X}_{34}\\ \mathbb{X}_{14}^T & \mathbb{X}_{24}^T & \mathbb{X}_{34}^T & \mathbb{X}_{44}\end{pmatrix} \tag{28}$$

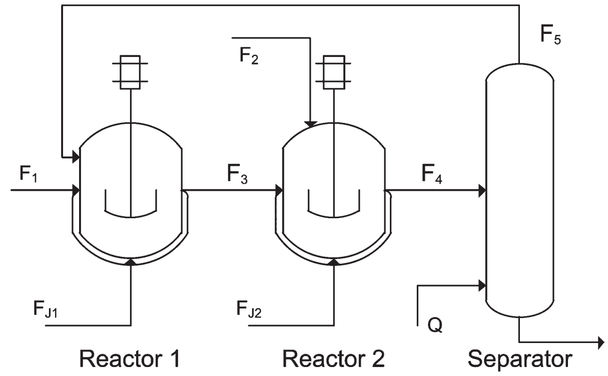

Figure 4. Reactor separator process.

$$\Upsilon_{12}(\zeta,\eta) = \begin{pmatrix} \mathcal{S}_d + F_p^T H_p^T \mathcal{R}_{Id} \\ \mathcal{R}_{dL}^T c \\ 0 \\ 0 \end{pmatrix} \tag{29}$$

$$\Upsilon_{13}(\zeta,\eta) = \begin{pmatrix} F_I^T \mathcal{S}_c^{rL^T} \hat{C} \\ \mathcal{Q}_c^{Lr} \hat{C} + \hat{D}_1^T \mathcal{Q}_c^{rr} \hat{C} \\ \hat{D}_2^T \mathcal{Q}_c^{rr} \hat{C} \\ \mathcal{S}_c^{rr^T} \hat{C} \end{pmatrix} \tag{30}$$

$$\Upsilon_{22}(\zeta,\eta) = \mathcal{R}_{dd} \tag{31}$$

$$\Upsilon_{23} = 0 \tag{32}$$

$$\Upsilon_{33} = \hat{C}^T \mathcal{Q}_c^{rr} \hat{C} \tag{33}$$

where

$$\begin{aligned}
\mathbb{X}_{11} &= \mathcal{Q} + \mathcal{S}_I H_p F_p + F_p^T H_p^T \mathcal{S}_I^T + F_p^T H_p^T \mathcal{R}_{II} H_p F_p + F_I^T \mathcal{R}_c^{LL} F_I \\
\mathbb{X}_{12} &= \mathcal{S}_L + H_p^T F_p^T \mathcal{R}_{IL} + F_I^T \left(\mathcal{S}_c^{LL} + \hat{D}_1^T \mathcal{S}_c^{rL} \right)^T \\
\mathbb{X}_{13} &= F_I^T \left(\hat{D}_2^T H_p^T \mathcal{S}_c^{rL} \right)^T \\
\mathbb{X}_{14} &= F_I^T \mathcal{R}_c^{LI} \\
\mathbb{X}_{22} &= \mathcal{R}_{LL} + \mathcal{Q}_c^{LL} + \mathcal{Q}_c^{LI} \hat{D}_1 + \hat{D}_1^T \mathcal{Q}_c^{LI^T} + \hat{D}_1^T \mathcal{Q}_c^{II} \hat{D}_1 \\
\mathbb{X}_{23} &= \mathcal{Q}_c^{LI} \hat{D}_2 H_p + \hat{D}_1^T \mathcal{Q}_c^{II} \hat{D}_2 H_p \\
\mathbb{X}_{24} &= \hat{D}_1^T \mathcal{S}_c^{II} + \mathcal{S}_c^{Lr} \\
\mathbb{X}_{33} &= \hat{D}_2^T H_c^T \mathcal{Q}_c^{II} H_c \hat{D}_2 \\
\mathbb{X}_{34} &= \hat{D}_2^T H_p^T \mathcal{S}_c^{II} \\
\mathbb{X}_{44} &= \mathcal{R}_c^{II}
\end{aligned} \tag{34}$$

Proof. See Appendix A. ∎

As the controller is not dissipative in the usual sense, the result above it valid for the output trajectory of the closed loop plant-wide system. However, as the disturbance enters the system through as an input to the individual processes, which are dissipative in the usual sense, it is valid for all disturbances.

Distributed Dissipative MPC

In this section, the distributed dissipative MPC algorithm is presented, along with results ensuring global stability, minimum performance bounds and recursive feasibility. The design of the MPC is a two step process. First, in an offline step, the required dissipative trajectory properties of the controllers are determined subject to the dissipativity of the individual processes, process network topology, and controller network topology such that the plant-wide system is stable and meets the minimum performance requirement. subsequently, a dissipative trajectory constraint is added to the online MPC algorithm of each controller to ensure the controller traces the dissipative trajectory determined in the first step for plant-wide stability and performance.

In this article, the controllers communicate simultaneously by sharing their closed-loop predictions (i.e., taking their control action and communication from other controllers into account) of their local processes output. This communication can be carried out iteratively within each sampling instant to improve predictions as the effects of other controllers actions are taken into account once they become available (from the previous iteration). At each iteration the constraint ensuring the controllers trace a dissipative trajectory (presented below) is satisfied, thus, if the iteration is terminated prematurely after an arbitrary number of iterations the controller will continue to trace a dissipative trajectory. This framework also allows the controllers to only communicate once per sampling instance by sending their closed-loop predictions from last sampling period. By doing this, the communication and computational load is reduced at the cost of possible loss of performance.

Online dissipativity constraint and feasibility

At any point in time, k, the QDF supply rate of the i-th controller is a function of its current inputs and outputs as well as its predicted future inputs and outputs. Thus, for a supply rate of order N for all sampling instances, t, such that $k < t \leq k + N$ the value of the the supply rate $Q_{\phi_c}(k)$ is not fully determined. Thus, the dissipative trajectory condition at time t may be written as

$$W_{t-(N+1)} + \sum_{i=t-N}^{t} Q_{\phi_c}(i) \geq 0 \tag{35}$$

This condition is stated as an LMI constraint in Proposition 2.

Proposition 2. *Given the supply rate of the i-th controller (which has been determined offline) induced by*

$$\tilde{\Theta}_i = \begin{pmatrix} \tilde{\mathcal{Q}}_{ci} & \tilde{\mathcal{S}}_{ci} \\ \tilde{\mathcal{S}}_{ci}^T & \tilde{\mathcal{R}}_{ci} \end{pmatrix} \tag{36}$$

in the extended input output space, this controller traces a dissipative trajectory at any point in time k, if the following linear matrix inequality is satisfied

$$\begin{pmatrix} -\chi^{-1} & \hat{y}_L(k) \\ \hat{y}_L^T(k) & \Omega + W_{k-N-1} \end{pmatrix} \geq 0 \tag{37}$$

where N is the order of the controller supply rate and $\hat{y}_L(k) = (y_L^T(k)\, y_L^T(k+1) \cdots y_L^T(k+N))^T$ *is the vector of predicted future controller local outputs at time k.* χ *is a function of the controller supply rate, representing the quadratic term of* $\hat{y}_L(k)$ *in the controller dissipation inequality. The form of* χ *varies with the order of the supply rate. For ease of exposition the case of a first order supply rate is shown as*

$$\chi = \begin{pmatrix} Q_{11} + S_{11}^L D_1^L + D_1^{L^T} R_{11}^{LL} D_1^L & Q_{10} + S_{11}^L D_2^L + D_1^{L^T} S_{01}^{L^T} + D_1^{L^T} R_{11}^{LL} D_2^L \\ \left(Q_{10} + S_{11}^L D_2^L + D_1^{L^T} S_{01}^{L^T} + D_1^{L^T} R_{11}^{LL} D_2^L \right)^T & Q_{00} + Q_{11} + S_{01}^L D_2^L + D_2^{L^T} R_{11}^{LL} D_2^L \end{pmatrix} \tag{38}$$

In this case $\tilde{\mathcal{Q}}_{ci} = \begin{pmatrix} \mathcal{Q}_{00} & \mathcal{Q}_{01} \\ \mathcal{Q}_{10} & \mathcal{Q}_{11} \end{pmatrix}$ *with* D_1^L and D_2^L *parameters of the local process model,* P_L

$$P_L : \left\{ \begin{pmatrix} x(k+1) \\ y(k) \end{pmatrix} = \begin{pmatrix} A & B_1 & B_2 & B_3 \\ C & D_1 & D_2 & D_3 \end{pmatrix} \begin{pmatrix} x(k) \\ u_c(k) \\ u_p(k) \\ d(k) \end{pmatrix} \right. \quad (39)$$

Ω *is a quadratic form defined in Eq.* 100.

Proof. See Appendix A. ■

Proposition 2 allows the formulation of the dissipative trajectory constraint in a convex manner. Given that in general $\phi_c(\zeta, \eta)$ is indefinite the recursive feasibility of the dissipative trajectory constraint may not be ensured, especially when constraints are put on the controller. The following result gives a sufficient condition for recursive feasibility in the presence of input constraints.

Proposition 3. *A sufficient condition for the dissipative trajectory condition given in Proposition 2 to be feasible at any time* $k > 0$ *in the unconstrained case or with hard input constraints is that the trajectory condition is originally feasible* ($W_0 \geq 0$)*, the constraint set* $u_c(k) \in \mathcal{U}$ *contains the origin and* $\Omega \geq 0$.

Proof. See Appendix A. ■

The above condition ensures recursive feasibility given feasibility of the control algorithm at $t = 0$ (note that this is trivially true if the system is at rest at $t = 0$). This does not impose any explicit constraints on the order of the controller supply rate. Initial feasibility may be easier to satisfy if the order of the controller supply rate is low (less stringent constraints on the trajectory of the controller). However, the higher the order of the controller and process supply rates the more information may be captured in the supply rates, allowing the controller to more effectively shape the dissipativity properties of the plant-wide system (and therefore plant-wide stability and performance properties).

This allows for commonly used box constraints (i.e., $\underline{u}_{c_i} \leq u_{c_i} \leq \overline{u}_{c_i}$) and ellipsoid constraints (*i.e.*, $u_{ci}^T F_i u_{c_i} \leq 1$). This result is reminiscent of those provided in Raff et al.[15] and Heath et al.[26] In Raff et al.[15] it is required that the control input, $y_c = u$, is always feasible to ensure a passive control is feasible, indeed this may be seen as a special case of our formulation. Similarly, in Heath et al,[26] stability is ensured by showing that the MPC algorithm satisfies a sector condition. This, however, requires that the origin remains feasible.

Plant-wide stability and performance conditions

Using the plant-wide dissipative trajectory formulation developed in section the main plant-wide stability condition can now be presented assuming that each controller satisfies the conditions in Proposition 2, and thus traces a dissipative trajectory. It should be noted that as the dissipativity of the controller is defined differently to the usual definition, the technicalities of this result are different to those in the literature (c.f. Tippett and Bao[18]).

Theorem 2. *Consider the plant-wide system shown in Figure 2 with n individual processes and controllers and process and controller network topology given by* H_p *and* H_c*, respectively. Assume that each controller traces a dissipative trajectory as in Definition 1 and that the individual processes are dissipative such that the plant-wide systems dissipativity properties are given by Lemma 1. The plant-wide system from external disturbances* (**d**) *to combined process and controller outputs* $(\mathbf{y_{pw}}^T = (\mathbf{y}^T\ \mathbf{y}_L))^T$ *is asymptotically stable if*

(1) *The overall process without interconnection,* $\tilde{\mathcal{P}}$*, has nonnegative storage. i.e.,* $\Psi(\zeta, \eta) \geq 0$

(2) $$\begin{pmatrix} \Upsilon_{11}(\zeta,\eta) & \Upsilon_{13}(\zeta,\eta) \\ \Upsilon_{13}^T(\zeta,\eta) & \Upsilon_{33}(\zeta,\eta) \end{pmatrix} < 0$$

where $\Upsilon^{11}(\zeta, \eta)$, $\Upsilon^{13}(\zeta, \eta)$, *and* $\Upsilon^{33}(\zeta, \eta)$ *are defined in Lemma 1.*

Proof. See Appendix A. ■

Remark 2. *The distributed MPC will stabilize the plant-wide system if the conditions in Theorem 2 are satisfied, even if iterative communication is stopped after an arbitrary number of iterations. As was mentioned at the beginning of this section, the controller is guaranteed to trace a dissipative trajectory if communication is halted after an arbitrary number of iterations as the dissipative trajectory constraint is satisfied every iteration.*

In Chen and Scherer[27] dissipativity is used to ensure minimum performance guarantees for model predictive control of single systems using supply rates of the form $(-I, 0, \gamma^2 I)$. In the following an analogous result ensuring minimum weighted norm bounds using dissipative trajectories and more general QDF supply rates is presented. This not only provides more detailed performance measures, but also provides a less conservative result as more general forms of dissipativity may be considered. The following lemma presents the plant-wide supply rate in a slightly different form to Lemma 1. The difference is that in Lemma 1 the nature of the controller communication (i.e. of the predicted process outputs) is explicitly put into the controller supply rate. In Lemma 2, the nature of controller communication is not specified. Thus, it is a slightly more general form of supply rate, and if the plant-wide system traces a dissipative trajectory with supply rate given in Lemma 1, then it will also trace a dissipative trajectory with supply rate given below.

Lemma 2 *Consider the interconnected system as shown in Figure 2 with n processes and controllers and process and controller network topology given by* H_p *and* H_c*, respectively. If the overall process is dissipative with respect to supply rate* Q_Φ *and storage function* Q_Ψ *and the controller traces a dissipative trajectory with supply rate* Q_Θ*. Then, for any disturbance, the plant-wide system from all disturbances* **d** *to all process output and controller output* $\mathbf{y_{pw}} = [\mathbf{y}^T, \mathbf{y_L}^T, \mathbf{y_r}^T]^T$ *satisfies*

$$\sum^{k} Q_\mu(k) \geq Q_\Psi(k) \quad (40)$$

for all $k \geq 0$*. That is to say it traces a dissipative trajectory with supply rate* Q_μ *where*

$$\mu(\zeta,\eta) = \begin{pmatrix} \Gamma_{11}(\zeta,\eta) & \Gamma_{12}(\zeta,\eta) \\ \Gamma_{12}^T(\zeta,\eta) & \Gamma_{22}(\zeta,\eta) \end{pmatrix} \quad (41)$$

$$\Gamma_{11}(\zeta,\eta) = \begin{pmatrix} \mathbb{W}_{11} & \mathbb{W}_{12} & \mathbb{W}_{13} \\ \mathbb{W}_{12}^T & \mathbb{W}_{22} & \mathbb{W}_{23} \\ \mathbb{W}_{13}^T & \mathbb{W}_{23}^T & \mathbb{W}_{33} \end{pmatrix} \quad (42)$$

$$\Gamma_{12}(\zeta,\eta) = \begin{pmatrix} \mathcal{S}_d + F_p^T H_p^T \mathcal{R}_{Id} \\ \mathcal{R}_{dL}^T \\ 0 \end{pmatrix} \quad (43)$$

$$\Gamma_{22}(\zeta,\eta) = \mathcal{R}_{dd} \tag{44}$$

and

$$\mathbb{W}_{11} = \mathcal{Q} + \mathcal{S}_I H_p F_p + F_p^T H_p^T \mathcal{S}_I^T + F_p^T H_p^T \mathcal{R}_{II} H_p F_p + F_I^T \mathcal{R}_c^{ll} F_I \tag{45}$$

$$\mathbb{W}_{12} = \mathcal{S}_L + F_p^T H_p^T \mathcal{R}_{IL} + F_I^T \mathcal{S}_c^{ll^T} \tag{46}$$

$$\mathbb{W}_{13} = \left(\mathcal{S}_c^{lr} + H_c^T \mathcal{R}_c^{lr}\right) F_I \tag{47}$$

$$\mathbb{W}_{22} = \mathcal{R}_{LL} + \mathcal{Q}_c^{ll} \tag{48}$$

$$\mathbb{W}_{23} = \mathcal{Q}_c^{lr} + \mathcal{S}_c^{lr} H_c \tag{49}$$

$$\mathbb{W}_{33} = \mathcal{Q}_c^{rr} + \mathcal{S}_c^{rr} H_c + H_c^T \mathcal{S}_c^{rr^T} + H_c^T \mathcal{R}_c^{rr} H_c \tag{50}$$

Proof. To aid readability only a sketch of the proof is given. The result is easily shown by adding the two dissipation inequalities, $Q_\Phi \geq Q_\Psi$ and $Q_\Theta \geq 0$. Then substituting in the interconnection relations to eliminate all variables except the external inputs and process and controller outputs.

The following result provides minimum performance bounds on the plant-wide system with distributed MPC. Loosely, it implies that for any external disturbance the weighted trajectory of $\mathbf{y_{pw}}$ is bounded by the disturbance. That is to say that the trajectory of the plant-wide system is constrained to be bounded by the disturbance.

Theorem 3. *Consider a plant-wide process $\tilde{\mathcal{P}}$ with a supply rate of $\Psi \geq 0$ with dissipative distributed MPCs (as described in this section). If the plant-wide system from external disturbances,* $\mathbf{d}$, *to combined process and controller outputs,* $\mathbf{y_{pw}} = (y^T\ y_L^T)^T$, *traces a dissipative trajectory with respect to supply rate* Q_μ, *and* $\mu = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_{22} \end{pmatrix}$ *with* $\Gamma_{11} < 0$, *then the minimum plant-wide performance level*

$$\|W\mathbf{y_{pw}}\| \leq \|\mathbf{d}\| \tag{51}$$

is guaranteed. Choosing

$$W(z) = \frac{1}{p(z)} \hat{\Gamma}_{11}^{\frac{1}{2}}(z)$$

with $\underline{\sigma}(W(j\omega)) \geq \frac{1}{\gamma}\ \forall\omega \in [0, 2\pi]$ *and a scalar* $p(\eta)$ *such that its coefficient column vector,* p, *satisfies* $p^T p \geq \max\ (\bar{\sigma}(\Gamma_{22} + \Gamma_{12}^T \hat{\Gamma}_{11}\ \Gamma_{12}),\ \bar{\sigma}(\Gamma_{12}^T\ \hat{\Gamma}_{11}\ \Gamma_{12}))$, *we have*

$$\|\mathbf{y_{pw}}\| \leq \gamma\|\mathbf{d}\| \tag{52}$$

where $\bar{\sigma}$ *and* $\underline{\sigma}$ *denote the maximum and minimum singular values respectively.*

Proof. See Appendix A.

Note that the above theorem guarantees the plant-wide stability and minimum (worst-case) performance bound if the controllers communicate at least once per sampling period. If the controllers are allowed to communicate iteratively within each sampling period, then the actual performance may be improved (as the controllers predictions of future process outputs are refined). However, such possible improvement of performance is not reflected by the worst case performance bound addressed in Theorem 3.

Distributed control design

Using the results above, both the offline problem to determining the system dissipativity and the online problem to carry out local optimization subject to global stability and minimum performance (in the form of required dissipative trajectory constraints) can be stated. These problems are formulated in the forms of LMIs which can be efficiently solved. The following LMI problem, evaluated offline, yields a supply rate for the distributed controllers. Which, if implemented as in Problem 2, will ensure recursive feasibility of the MPC algorithm(s) and plant-wide internal stability.

Problem 1. *Consider a plant-wide system with n individual processes and controllers with process and controller network topologies defined by H_p and H_c as in Section. Given $\psi_i \geq 0\ \forall i \in [1,n]$, the cost functions of the i-th controller. Find a set of supply rates parameterized by Q_i, S_i, R_i, Q_{c_i}, S_{c_i}, and R_{c_i} such that ψ_i is the storage function of the i-th process and the following LMI constraints are satisfied*

$$\begin{pmatrix} \mathbb{T}_{11_i} & \mathbb{T}_{12_i} \\ \mathbb{T}_{12_i}^T & \mathbb{T}_{22_i} \end{pmatrix} \geq 0 \quad \forall i \tag{53}$$

(*Dissipativity of process i*)

$$\Omega_i \geq 0 \quad \forall i \tag{54}$$

$$\chi_i < 0 \quad \forall i \tag{55}$$

(*Feasibility of controller i supply rate*)

$$\begin{pmatrix} \tilde{\Upsilon}_{11} & \tilde{\Upsilon}_{13} \\ \tilde{\Upsilon}_{13}^T & \tilde{\Upsilon}_{33} \end{pmatrix} < 0 \tag{56}$$

(*Internal stability of plant-wide process*)

$$\tilde{\Gamma}_{11} \leq -\tilde{N}^T \tilde{N} \tag{57}$$

$$\tilde{\Gamma}_{12} = 0 \tag{58}$$

$$\tilde{\Gamma}_{22} \geq \tilde{d}^T \tilde{d} \tag{59}$$

(*Minimum performance guarantee*)

Along with $\psi_i \geq 0\ \forall i$, Condition (53) ensures the dissipativity of the processes as per Proposition (1). Conditions (54) and (55) ensure the recursive feasibility of the online MPC algorithm, while Condition (56) ensures the stability of the plant-wide system as per Theorem (2). Finally conditions (57) to (59) ensure a minimum plant-wide performance level as per Theorem 3 with $W(s) = \frac{1}{d(s)} N(s)$ with a scalar $d(s)$.

The solution of Problem 1 yields a set of controller supply rates that ensure the recursive feasibility, plant-wide stability and minimum performance bounds. Once this is solved the online MPC algorithm (for each distributed controller) can then be implemented. This is stated below, note that it is assumed that iteration is arbitrarily halted after $I_{\max}$ iterations.

Problem 2. *The algorithm for the i-th controller with hard input constraints and soft output constraints has local output, $\hat{y_L}(k)$, which is the minimizer of the i-th controllers storage function, ϕ_i. At any time step k it is as follows:*

1. *If $k = 0$ set $\hat{u_r}(k) = 0$ (the input from remote controllers). If $k > 0$ and this is the first iteration (of this time step) set $\hat{u_r}(k)$ as the last one received last time step shifted forward one time step, $\hat{u_r}(k) = \sigma\hat{u_r}(k-1)$. Otherwise set $\hat{u_r}(k)$ that which is received from the other controllers.*

2. *Optimize local input by the following LMI problem*

$$\hat{y}_L(k) = \arg\min\ \alpha \tag{60}$$

subject to

$$\begin{pmatrix} \left[\hat{D}_1^T\psi_{11}\hat{D}_1 + \hat{D}_1^T\psi_{12} + \psi_{12}^T\hat{D}_1 + \psi_{22}\right]^{-1} & \hat{y}_L(k) \\ \hat{y}_L^T(k) & \alpha + \Lambda + w_y\epsilon \end{pmatrix} \geq 0 \tag{61}$$

$$\begin{pmatrix} -\chi^{-1} & \hat{y}_L(k) \\ \hat{y}_L^T(k) & \Omega + W_{k-n-1} \end{pmatrix} \geq 0 \tag{62}$$

$$-\chi > 0 \tag{63}$$

$$\hat{y}_L(k) \in \mathcal{U} \tag{64}$$

$$\begin{pmatrix} P^{-1} & \hat{C}x(k) + \hat{D}_1\hat{y}_L + \hat{D}_2\hat{u}_r \\ \hat{C}x(k) + \left(\hat{D}_1\hat{y}_L + \hat{D}_2\hat{u}_r\right)^T & 1+\epsilon \end{pmatrix} \geq 0 \tag{65}$$

$$\epsilon \geq 0 \tag{66}$$

where $\mathcal{U}$ is any convex set containing the origin such that the constraint $\hat{y}_L(k) \in \mathcal{U}$ may be written as a LMI, the (ellipsoid) output constraints are $\hat{y}^T(k)P\hat{y}(k) \leq 1 + \epsilon$ with $P > 0$, $\epsilon \geq 0$ and

$$\begin{aligned} \Lambda = & -x^T(k)\hat{C}^T\psi_{11}\hat{C}x(k) - 2x^T(k)\hat{C}^T\left(\psi_{11}\hat{D}_1 + \psi_{12}\right)\hat{y}_L \\ & - 2x^T(k)\hat{C}^T\left(\psi_{11}\hat{D}_2 + \psi_{14}\right)\hat{u}_r \\ & - 2\hat{y}_L^T\left(\hat{D}_1^T\psi_{11}\hat{D}_2 + \psi_{24} + \psi_{12}^T\hat{D}_2^T + \hat{D}_1^T\psi_{14}\right)\hat{u}_r \\ & - \hat{u}_r^T\left(\hat{D}_2^T\psi_{11}\hat{D}_2 + \hat{D}_2^T\psi_{14} + \psi_{14}^T\hat{D}_2 + \psi_{44}\right)\hat{u}_r \end{aligned} \tag{67}$$

3. *Calculate, and send to remote controller(s) predicted local process output*

$$\hat{y}_i = \hat{C}x_i(k) + \hat{D}_{1_i}\hat{u}_{L_i} + \hat{D}_{2_i}\hat{u}_{p_i} \tag{68}$$

4. *If number of iterations is greater or equal to $I_{\max}$ cease iteration and apply optimal output in Step 2. Otherwise return to step 1.*

Proof. See Appendix A. ■

Remark 3. *The communication between controllers in Step 3 of Problem 2 is not to all controllers. Instead it is only to those controllers with the i-th controller communicates to as described by the controller network topology and captured by H_c.*

Remark 4. *Problem 2 is guaranteed to be recursively feasible with hard input and soft output constraints if the condition in Proposition 3 is satisfied. This represents an important class of industrial problems. Additional constraints such as input increment constraints can also be added to the problem in an obvious way. Note, however, that to ensure recursive feasibility it may be necessary to implement these as soft constraints.*

Illustrative Example

A comparison of distributed and decentralized control using the proposed framework is illustrated by a simple reactor separator process which contains two CSTRs in series followed by a flash separator with recycle of the top product of the separator (as shown in Figure 4). Similar processes have been studied previously in Liu et al.[10] and Christofides et al.[12] The first order reactions $A \rightarrow B \rightarrow C$ occur in the reactors with B the desired product, it is assumed no reaction occurs in the separator. The manipulated variables are the flow rates in the reactor jackets and heat input into the separator. The disturbances are in the concentration of feed F_1 and temperature of feed F_2. The measured outputs of the reactors are the outlet temperature, it is assumed that the concentrations as well as temperature of the tops product of the separator are measured. Under assumptions of perfect mixing and constant physical properties the system equations from the mass and energy balances are given below

Reactor 1

$$\dot{x}_{A1} = \frac{F_1}{V_1}(x_{A_{F1}} - x_{A1}) + \frac{F_5}{V_1}(x_{Ar} - x_{A1}) - k_1 e^{\frac{-E_1}{RT_1}} x_{A1} \tag{69}$$

$$\dot{x}_{B1} = \frac{F_1}{V_1}(x_{B_{F1}} - x_{B1}) + \frac{F_5}{V_1}(x_{Br} - x_{B1}) + k_1 e^{\frac{-E_1}{RT_1}} x_{A1} - k_2 e^{\frac{-E_2}{RT_1}} x_{B1} \tag{70}$$

$$\begin{aligned} \dot{T}_1 = & \frac{F_1}{V_1}(T_{F_1} - T_1) + \frac{F_5}{V_1}(T_3 - T_1) - \frac{\Delta H_1}{C_p} k_1 e^{\frac{-E_1}{RT_1}} x_{A1} \\ & - \frac{\Delta H_2}{C_p} k_2 e^{\frac{-E_2}{RT_1}} x_{B1} + \frac{UA_1(T_{J1} - T_1)}{\rho C_p V_1} \end{aligned} \tag{71}$$

$$\dot{T}_{J1} = \frac{F_{J1}}{V_{J1}}(T_{J1_{in}} - T_{J1}) + \frac{UA_1(T_1 - T_{J1})}{\rho_J C_{pJ} V_{J1}} \tag{72}$$

Reactor 2

$$\dot{x}_{A2} = \frac{F_3}{V_2}(x_{A1} - x_{A2}) + \frac{F_2}{V_2}(x_{A_{F2}} - x_{A2}) - k_1 e^{\frac{-E_1}{RT_2}} x_{A2} \tag{73}$$

$$\dot{x}_{B2} = \frac{F_3}{V_2}(x_{B1} - x_{B2}) + \frac{F_2}{V_2}(x_{B_{F2}} - x_{B2}) + k_1 e^{\frac{-E_1}{RT_2}} x_{A2} - k_2 e^{\frac{-E_2}{RT_2}} x_{B2} \tag{74}$$

$$\begin{aligned} \dot{T}_2 = & \frac{F_3}{V_2}(T_1 - T_2) + \frac{F_2}{V_2}(T_{F2} - T_2) - \frac{\Delta H_1}{C_p} k_1 e^{\frac{-E_1}{RT_2}} x_{A2} \\ & - \frac{\Delta H_2}{C_p} k_2 e^{\frac{-E_2}{RT_2}} x_{B2} + \frac{UA_2(T_{J2} - T_2)}{\rho C_p V_2} \end{aligned} \tag{75}$$

$$\dot{T}_{J2} = \frac{F_{J2}}{V_{J2}}(T_{J2_{in}} - T_{J2}) + \frac{UA_2(T_2 - T_{J2})}{\rho_J C_{pJ} V_{J2}} \tag{76}$$

separator

$$\dot{x}_{A3} = \frac{F_4}{V_3}(x_{A2} - x_{A3}) - \frac{F_5 + F_p}{V_3}(X_{Ar} - X_{A3}) \tag{77}$$

$$\dot{x}_{B3} = \frac{F_4}{V_3}(x_{B2} - x_{B3}) - \frac{F_5 + F_p}{V_3}(X_{Br} - X_{B3}) \tag{78}$$

$$\dot{T}_3 = \frac{F_4}{V_3}(T_2 - T_3) + \frac{Q_3}{\rho C_p V_3} \tag{79}$$

Table 1. Process Parameters

Parameters	Values	Units	Description
F_1	5.04	$m^3\ hr^{-1}$	External feed flow rate
$x_{A_{F1}}$	1	-	Mass fraction of A in F_1
$x_{B_{F1}}$	0	-	Mass fraction of B in F_1
F_2	5.04	$m^3\ hr^{-1}$	Fresh feed flow rate
$x_{A_{F21}}$	1	-	Mass fraction of A in F_2
$x_{B_{F2}}$	0	-	Mass fraction of B in F_2
V_1	1	m^3	Volume of reactor 1
V_2	0.5	m^3	Volume of reactor 2
V_3	1	m^3	Volume of separator
UA_1	2509.8	WK^{-1}	Heat transfer coefficient
UA_2	2774	WK^{-1}	Heat transfer coefficient
ρ	1000	$kg\ m^{-3}$	Density in units
ρ_J	1000	$kg\ m^{-3}$	Density in jacket
C_p	4.2	$kJ\ kg^{-1}\ K^{-1}$	Heat capacity
C_{pJ}	4.2	$kJ\ kg^{-1}\ K^{-1}$	Heat capacity in jacket
k_1	9.97×10^6	hr^{-1}	Rate constant of $A \rightarrow B$
k_2	9×10^6	hr^{-1}	Rate constant of $B \rightarrow C$
E_1	5×10^4	$J\ mol^{-1}$	Activation energy of $A \rightarrow B$
E_2	6×10^4	$J\ mol^{-1}$	Activation energy of $B \rightarrow C$
R	8.314	$J\ mol^{-1}\ K^{-1}$	Gas constant
ΔH_1	-6×10^4	$J\ mol^{-1}$	Heat of reaction
ΔH_2	-7×10^4	$J\ mol^{-1}$	Heat of reaction
α_A	3.5	-	Relative volatility of A
α_B	1	-	Relative volatility of B
α_C	0.5	-	Relative volatility of C

where it has been assumed that all units have constant hold up and all species have constant relative volatility. The following relations define the separator overhead concentrations

$$x_{Ar} = \frac{\alpha_A x_{A3}}{\alpha_A x_{A3} + \alpha_B x_{B3} + \alpha_C x_{C3}} \tag{80}$$

$$x_{Br} = \frac{\alpha_B x_{B3}}{\alpha_A x_{A3} + \alpha_B x_{B3} + \alpha_C x_{C3}} \tag{81}$$

$$x_{Cr} = \frac{\alpha_C x_{C3}}{\alpha_A x_{A3} + \alpha_B x_{B3} + \alpha_C x_{C3}} \tag{82}$$

The process parameters are shown in Table 1.

The equations are linearized around the operating point in Table 2. A dissipative distributed model predictive controller is designed using the proposed approach. As per the network decomposition in section, we decompose the plant-wide system along the physical layout of the system. That is, we treat the two reactors and the separator as individual subsystems and design one controller for each. The plant-wide control objective is to minimize the plant-wide extended $\mathcal{L}_2$ gain as in Theorem 3 with respect to disturbances in the concentration of the feed to the first reactor, F_1, and the temperature of the feed to the second reactor, F_2. This is subject to input constraints $|F_{J1}^*| \leq 0.5$, $|F_{J2}^*| \leq 0.5$ and $|Q_3^*| \leq 0.65$ where the '*' denotes normalized deviation from steady state.

Table 2. Process Operating Point

States	Operating Point	Units	Description
x_{A1}	0.383	–	Mass fraction of A in reactor 1
x_{B1}	0.581	–	Mass fraction of B in reactor 1
T_1	447.8	K	Temperature in reactor 1
T_{J1}	498	K	Temperature in reactor 1 jacket
x_{A2}	0.391	–	Mass fraction of A in reactor 2
x_{B2}	0.572	–	Mass fraction of B in reactor 2
T_2	444.6	K	Temperature in reactor 2
T_{J2}	503	K	Temperature in reactor 2 jacket
x_{A3}	0.172	–	Mass fraction of A in separator
x_{B3}	0.748	–	Mass fraction of B in separator
T_3	449.6	K	Temperature in separator

As discussed in the previous sections, a set of supply rates and storage function pairs for each subsystem and supply rate for each controller is determined to achieve this plant-wide aim. The storage functions of each subsystem are then used as the cost functions (ignoring the terms associated with the unmeasured disturbance) for their local controllers.

The matrices which induce the ith process storage function and ith controller cost function, $\psi_i(\zeta, \eta)$, are given below

$$\psi_1(\zeta,\eta) = \begin{pmatrix} 10I_2 + 10I_2\zeta + 10I_2\eta + 10I_2\zeta\eta & 0_{6\times14} \\ 0_{14\times6} & 5I_{14} + 5I_{14}\zeta + 5I_{14}\eta + 5I_{14}\zeta\eta \end{pmatrix} \tag{83}$$

$$\psi_2(\zeta,\eta) = \begin{pmatrix} 10I_2 + 10I_2\zeta + 10I_2\eta + 10I_2\zeta\eta & 0_{6\times14} \\ 0_{14\times6} & 5I_{14} + 5I_{14}\zeta + 5I_{14}\eta + 5I_{14}\zeta\eta \end{pmatrix} \tag{84}$$

$$\psi_3(\zeta,\eta) = \begin{pmatrix} I_2 + I_2\zeta + I_2\eta + I_2\zeta\eta & 0_{6\times8} \\ 0_{8\times6} & 0.1I_8 + 0.1I_8\zeta + 0.1I_8\eta + 0.1I_8\zeta\eta \end{pmatrix} \tag{85}$$

Pulse disturbances are introduced to the system in the form of a 5% increase in the mass fraction of A in the feed F_1 from 0.1 to 0.6 hr (and a pulse disturbance in the mass fraction of B) and a pulse disturbance in the temperature of flow F_2 of 10 K from time 2 to 4 hr. The sampling period for the simulation was 2 min, communication between controllers was halted after 1 iteration with a prediction horizon of three steps, the process and controller outputs for these simulations are shown in Figures 5 through 10. Note that the simulations are carried out using the linearized models of the processes.

The above results were compared with the decentralized case as shown in Figures 11 through 16. A dissipative decentralized MPC was designed with the controller network topology defined by $H_c = 0$ as per Remark 1. It can be seen that the distributed controller performs better than the decentralized controller with faster response and smaller overall error. The sum of the cost functions for the three controllers in the distributed case is 3.15×10^2, this is compared to 4.73×10^2 for the decentralised case illustrating a significant improvement in the distributed case. It is expected that increasing the number of iterations of controller communication would further improve performance in the distributed case. However, at the cost of increasing computational and communication load.

Discussion and Conclusions

Dissipative systems theory has been used to facilitate a novel plant-wide distributed MPC strategy. A dissipativity based approach is attractive as it allows for chemical process networks to be decomposed into individual subsystems, dissipativity properties of the subsystems determined and then the dissipativity properties of the plant-wide system calculated as a linear combination of that of the subsystems after substituting in the interconnection topology. Complicated interconnection structures, which are common in process control due to material recycle and heat integration, may then be easily handled. As the interconnection topology,

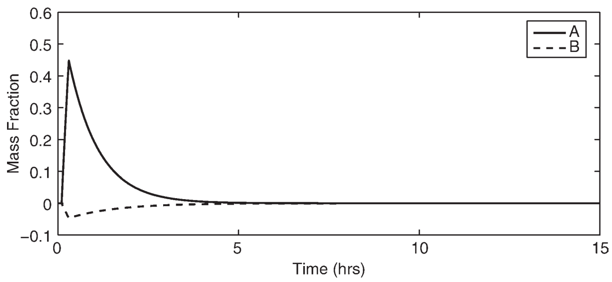

Figure 5. Reactor 1 concentrations.

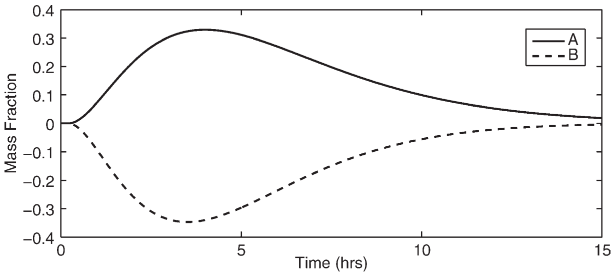

Figure 6. Reactor 2 concentrations.

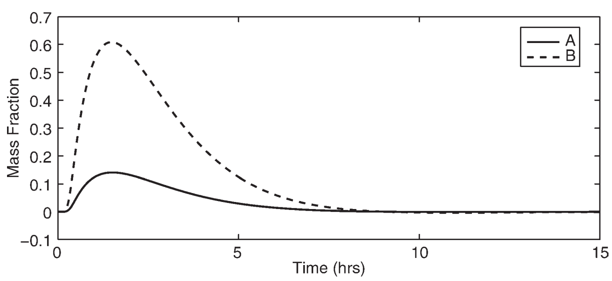

Figure 7. Separator concentrations.

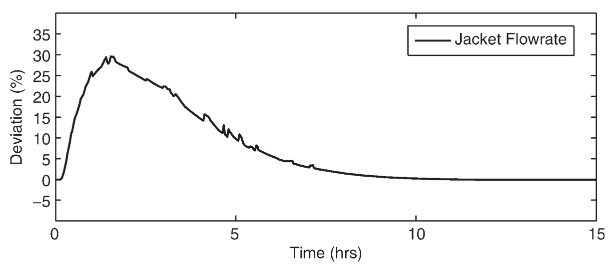

Figure 8. Controller 1 output.

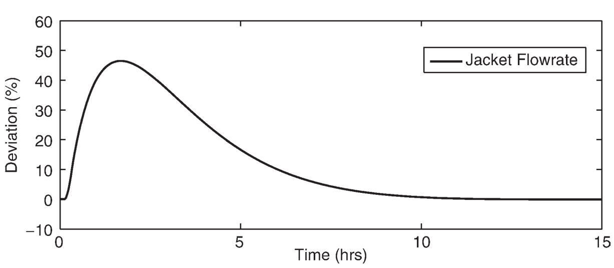

Figure 9. Controller 2 output.

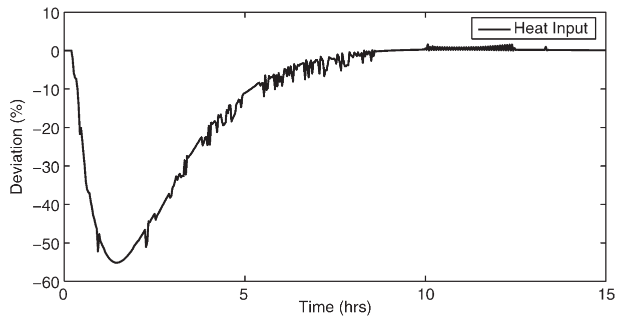

Figure 10. Controller 3 output.

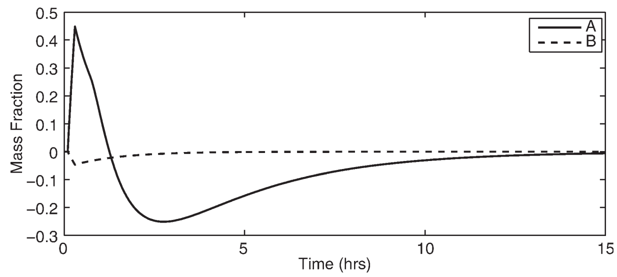

Figure 11. Reactor 1 concentrations.

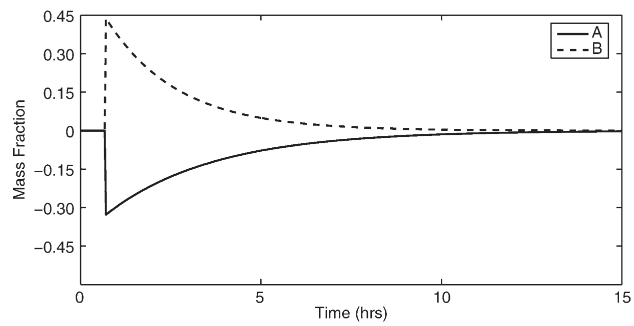

Figure 12. Reactor 2 concentrations.

both in the process and controller networks, is captured in the plant-wide dissipativity properties this allows a dissipativity based approach to account for interaction effects.

As both the plant-wide stability and performance conditions are encoded in the plant-wide dissipativity, the dissipative systems framework allows for a decomposition and distribution of these requirements amongst individual controllers. It also allows for different controller structures to be treated in a unified framework. For example, distributed and

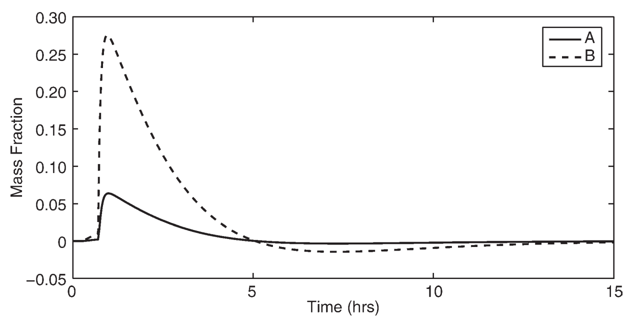

Figure 13. Separator concentrations.

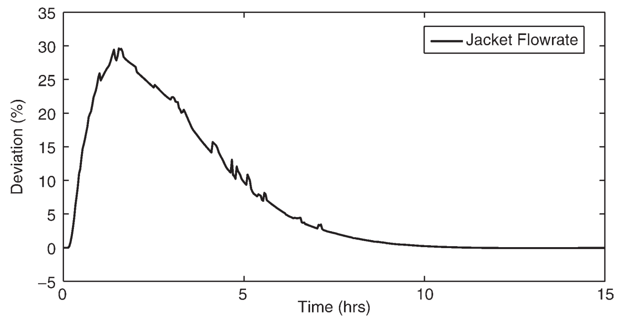

Figure 14. Controller 1 output.

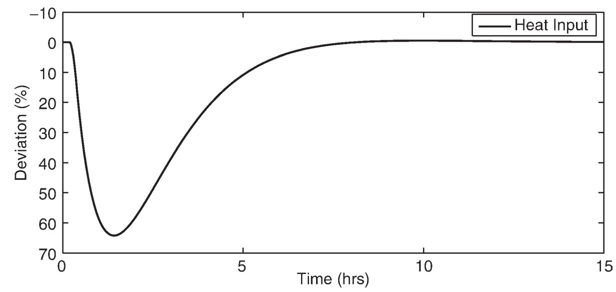

Figure 16. Controller 3 output.

decentralized control may both be considered. The form of the plant-wide supply rates varies with the controller network topology and therefore changing the plant-wide performance that can be obtained.

This provides a scalable framework for the control of large-scale systems where interaction effects are explicitly taken into account in the the stability and performance criteria. Communication in the distributed control scheme presented improves the controllers predictions by taking into account interaction effects and, importantly, allows for a trading of stability an performance among subsystems through their dissipativity properties. It is important to note that the dissipative trajectory condition placed on the MPC, being a property of the trajectory the controller follows, is conceptually different to the usual definition of dissipativity. As such, the stability and performance results are different to those available in the literature (for example, in the special case of (Q, S, R) dissipativity), even if they are cosmetically similar. This different version of dissipativity, along with associated plant-wide stability and performance results, are among the key contributions of this article.

Dynamic supply rates have been used as they provide less conservative results than constant supply rates, which has been a disadvantage of dissipativity based approaches in the past. The additional information captured, and reduction in conservativeness facilitates an approach to plant-wide dissipativity based on the interconnection of process and controllers networks and a supply rate centric control design methodology. More detailed performance criteria in the form of weighted norm bounds conditions are also formulated in a natural way using dynamic supply rates.

This approach has the advantage of allowing the controllers to communicate simultaneously not sequentially, thus not applying undue burden on any one controller. The proposed approach gains an advantage of scalability as the iterative optimization/controller communication is theoretically not required, compared to some cooperative approaches where conditions on the convergence of iteration optimizations and the constraints on communication bandwidth may make frequent iterations impractical for large scale problems. A scheme whereby the controllers communicate only once per sampling period by sending their predictions from the last step (similar to Jia and Krogh[8]) can be applied. An iterative communication scheme can also be implemented in the proposed approach with stability of the plant-wide system being assured even if the communication is halted after an arbitrary number of iterates. In this article, global performance objectives are encoded in the dissipativity of the plant-wide system. This has been done using quadratic differential forms in Tippett and Bao (submitted) for the continuous time case. This provides a mechanism for (minimum) global performance objectives to be enforced. In this framework communication between controllers not only increases the accuracy of controller predictions by taking interaction effects into account, but also allows for stability and performance to be traded amongst subsystems, thus decreasing the conservativeness of these conditions. An advantage of this approach is that if interactions between subsystems are beneficial this will be captured, and made use of, in the dissipativity, not simply canceled. This is akin to the idea of compensation by excessive input feedforward, and output feedback, passivity.[28]

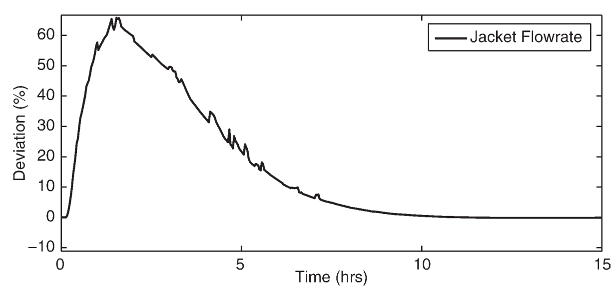

Figure 15. Controller 2 output.

The cost function for each controller is used as storage function for each associated process, and a stabilizing set of supply rates is then found offline, in what amounts to a LMI problem. The dissipativity condition is then added as an additional, convex, constraint to the online optimization algorithm. It is expected that computing the process and controller dissipativity online would allow for improved performance, especially for nonlinear systems. However, this becomes a nonconvex problem.

An extension of this work is the asynchronous, or multirate, control case. To make use of the significant separation of time scales both between, and within, unit operations that is often present in chemical processing applications. It would be expected that significant reductions in computation time could be realized in this way. Another extension of this work is to nonlinear systems. A difficulty with this, however, is that in general it is difficult to determine the dissipativity of a nonlinear system. However, recent research has demonstrated a fundamental link between the thermodynamic laws underpinning process systems and passivity, see for example Alonso and Ydstie,[29] Hangos et al.[30] and Bao et al.[31] This shows promise for extending this research to nonlinear chemical process control applications.

Finally, the optimization of the controller network topology and the identification of the most influential interconnections is of interest. However, this may be quite problem specific as the cost associated with communication may vary.

Acknowledgments

This work is supported by the Australian Research Council Discovery Project DP1093045. The first author would like to acknowledge the financial support of the Australian Postgraduate Award scholarship as well as the Engineering Supplementary Award and UNSW Excellence Awards.

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Appendix A

Proofs of theoretical results

Proof of Lemma 1. Assume the controller traces a dissipative trajectory

$$\sum_{i=0}^{k} Q_\Theta \geq 0 \quad \forall k \geq 0 \tag{A1}$$

From the dissipativity of the process

$$\sum_{i=0}^{k} Q_\Phi \geq Q_\Psi \quad \forall k \geq 0 \tag{A2}$$

is true for any allowable inputs and outputs of the process. Therefore, for the plant-wide system, we have

$$\sum_{i=0}^{k} Q_\Theta + Q_\Phi \geq Q_\Psi \geq 0 \quad \forall k \geq 0 \tag{A3}$$

Thus

$$\sum_{i=0}^{k} Q_\Theta + Q_\Phi \geq 0 \quad \forall k \geq 0 \tag{A4}$$

The above condition, by Definition 1, implies that the plant-wide system traces a dissipative trajectory with supply rate $Q_\Theta + Q_\Phi$. Note that this is satisfied for any allowable disturbance entering the process as the dissipativity of the processes is defined based on their input/output spaces, not trajectory. The controllers communicate by exchanging their predictions of the output of their own local process, that is

$$\hat{u}_r(k) = \hat{C}x(k) + \hat{D}_1\hat{y}_L(k) + \hat{D}_2 H_p \tilde{y}' \tag{A5}$$

where $\hat{D_1}$ and $\hat{D_2}$ are defined analogously to (14), after partitioning $B = (B_1\ B_2\ B_3)$ and $D = (D_1\ D_2\ D_3)$ (as in (21)) to separate the effects of the local control and interconnecting inputs from other units. Substituting this and the interconnection relations between variables in Figure 2 into the aforementioned overall supply rate and rearranging, the overall supply rate can be written as $Q_\mu(\mathbf{w})$ defined in (27) where $\mathbf{w} = (y^T\ y_L^T\ \tilde{y}^T\ \tilde{y}'^{\,T}\ d^T\ x^T(k))^T$ with $\mu(\zeta, \eta)$ defined as above. Where $\tilde{y} = u_R$ is the predicted process output and $\tilde{y}'$ is the predicted process output from the previous iteration (of controller communication).

Proof of Proposition 2. Permute the coefficient matrix of $\phi_c(\zeta, \eta)$ conformally with the controller inputs and outputs to be $P^T\tilde{\phi}_c P = \begin{pmatrix} \tilde{Q}_c & \tilde{S}_c \\ \tilde{S}_c^T & \tilde{R}_c \end{pmatrix}$. In the extended input–output space the controller traces a dissipative trajectory if at all times k

$$\sum^k \begin{pmatrix} \hat{y}_L \\ \hat{y}_R \end{pmatrix}^T \begin{pmatrix} \hat{Q}_c^{LL} & \hat{Q}_c^{LR} \\ \hat{Q}_c^{LR^T} & \hat{Q}_c^{RR} \end{pmatrix} \begin{pmatrix} \hat{y}_L \\ \hat{y}_R \end{pmatrix} + 2\begin{pmatrix} \hat{y}_L \\ \hat{y}_R \end{pmatrix}^T \begin{pmatrix} \hat{S}_c^{LL} & \hat{S}_c^{LR} \\ \hat{S}_c^{RL} & \hat{S}_c^{RR} \end{pmatrix} \begin{pmatrix} \hat{u}_L \\ \hat{u}_R \end{pmatrix} + \begin{pmatrix} \hat{u}_L \\ \hat{u}_R \end{pmatrix}^T \begin{pmatrix} \hat{R}_c^{LL} & \hat{R}_c^{LR} \\ \hat{R}_c^{LR^T} & \hat{R}_c^{RR} \end{pmatrix} \begin{pmatrix} \hat{u}_L \\ \hat{u}_R \end{pmatrix} \geq 0 \tag{A6}$$

For an Nth order supply rate this condition becomes

$$W_{k-N-1} + Q_{\tilde{\phi}_c}(k-N) + Q_{\tilde{\phi}_c}(k-N+1) + \cdots + Q_{\tilde{\phi}_c}(k) \geq 0 \tag{A7}$$

where W_{k-N-1} is a constant, the sum of all previous values of the controller supply rate and the other terms relate to the current and recent values of the controller supply rate which are decision variables. Using $\hat{u}_R = \hat{y}_L$, $Q_{\tilde{\phi}_c}(k)$ becomes

$$y_L^T(k) Q_c^{LL} y_L(k) + 2y_L^T(k)\begin{pmatrix} S_c^{LL} + \hat{Q}_c^{LR} & S_c^{LR} \end{pmatrix}\begin{pmatrix} u_L(k) \\ u_R(k) \end{pmatrix} + \begin{pmatrix} u_L(k) \\ u_R(k) \end{pmatrix}^T \begin{pmatrix} R_c^{LL} + Q_c^{RR} + S_c^{RL} + S_c^{RL^T} & R_c^{LR} + S_c^{RR} \\ R_c^{LR^T} + S_c^{RR^T} & \hat{R}_c^{RR} \end{pmatrix}\begin{pmatrix} u_L(k) \\ u_R(k) \end{pmatrix} \geq 0 \tag{A8}$$

Let $Q_{LL} = Q$, $(S_{LL} + Q_{LR}\ S_{LR}) = (S^L\ S^R)$ and

$$\begin{pmatrix} R_c^{LL} + Q_c^{RR} + S_c^{RL} + S_c^{RL^T} & R_c^{LR} + S_c^{RR} \\ R_c^{LR^T} + S_c^{RR^T} & \hat{R}_c^{RR} \end{pmatrix} = \begin{pmatrix} R^{LL} & R^{LR} \\ R^{LR^T} & R^{RR} \end{pmatrix}$$

the controller dissipativity condition, Eq. 92, may then be written as

$$W_{k-n-1} + \begin{pmatrix} \hat{y}_L(k) \\ \vdots \\ \hat{y}_L(k-N) \\ \hat{u}_L(k) \\ \vdots \\ \hat{u}_L(k-n) \\ \hat{u}_R(k) \\ \vdots \\ \hat{u}_R(k-N) \end{pmatrix}^T \begin{pmatrix} \mathrm{diag}Q & \mathrm{diag}S^L & \mathrm{diag}S^R \\ \mathrm{diag}S^{L^T} & \mathrm{diag}R^{LL} & \mathrm{diag}R^{LR} \\ \mathrm{diag}S^{R^T} & \mathrm{diag}R^{LR^T} & \mathrm{diag}R^{RR} \end{pmatrix} \begin{pmatrix} \hat{y}_L(k) \\ \vdots \\ \hat{y}_L(k-N) \\ \hat{u}_L(k) \\ \vdots \\ \hat{u}_L(k-N) \\ \hat{u}_R(k) \\ \vdots \\ \hat{u}_R(k-N) \end{pmatrix} \geq 0 \tag{A9}$$

These vectors "overlap" in the sense that (assuming a supply rate of at least order one) $\hat{y}_L(k-1)$ and $\hat{y}_L(k)$ are both functions of $y_L(k)$, with the same relationship holding for $\hat{u}_L$ and $\hat{u}_R$. Once these future steps are realized, the predicted values are equated with the actual values. For ease of exposition this is shown below in the original, not extended, variable space (at an arbitrary time step, k) for the case of a first order supply rate (although it is analogous for higher orders)

$$W_{k-2} + \begin{pmatrix} y_L(k+1) \\ y_L(k) \\ y_L(k+1) \\ u_L(k+1) \\ u_L(k) \\ u_L(k-1) \\ u_R(k+1) \\ u_R(k) \\ u_R(k-1) \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{10} & 0 & S_{11}^L & S_{10}^L & 0 & S_{11}^R & S_{10}^R & 0 \\ * & Q_{00}+Q_{11} & Q_{10} & S^L01 & S_{00}^L + S_{11}^L & S_{10}^L & S_{01}^R & S_{00}^R + S_{11}^R & S_{10}^R \\ * & * & Q_{00} & 0 & S_{01}^L & S_{00}^L & 0 & S_{01}^R & S_{00}^R \\ * & * & * & R_{11}^{LL} & R_{10}^{LL} & 0 & R_{11}^{LR} & R_{10}^{LL} & 0 \\ * & * & * & * & R_{00}^{LL} + R_{11}^{LL} & R_{10}^{LL} & R_{01}^{LR} & R_{00}^{LR} + R_{11}^{LR} & R_{10}^{LR} \\ * & * & * & * & * & R_{00}^{LL} & 0 & R_{01}^{LR} & R_{00}^{LR} \\ * & * & * & * & * & * & R_{11}^{RR} & R_{10}^{RR} & 0 \\ * & * & * & * & * & * & * & R_{00}^{RR} + R_{11}^{RR} & R_{10}^{RR} \\ * & * & * & * & * & * & * & * & R_{00}^{RR} \end{pmatrix} \begin{pmatrix} y_L(k+1) \\ y_L(k) \\ y_L(k+1) \\ u_L(k+1) \\ u_L(k) \\ u_L(k-1) \\ u_R(k+1) \\ u_R(k) \\ u_R(k-1) \end{pmatrix} \geq 0 \tag{A10}$$

where vectors at time $k+1$ are predicted values of the local control output, local process output and input from remote controllers. Eliminate $\tilde{u}_L(k+1)$ using the model of the (local) process

$$\tilde{u}_L(k+1) = \hat{C}x(k) + D_1^L \tilde{y}_L(k+1) + D_2^L y_L(k) + D_1^R \tilde{u}'_R(k+1) + D_2^R \tilde{u}'_R(k) \tag{A11}$$

where the $\tilde{u}'_R$ denotes the prediction of remote process outputs from the last iteration of communication.

$$W_{k-2} + \begin{pmatrix} \hat{C}x(k) \\ y_L(k+1) \\ y_L(k) \\ y_L(k-1) \\ u_L(k) \\ u_L(k-1) \\ u_R(k+1) \\ u_R(k) \\ u_R(k-1) \\ u'_R(k+1) \\ u'_R(k) \end{pmatrix}^T \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} & Z_{18} & Z_{19} & Z_{110} & Z_{111} \\ * & Z_{22} & Z_{23} & Z_{24} & Z_{25} & Z_{26} & Z_{27} & Z_{28} & Z_{29} & Z_{210} & Z_{211} \\ * & * & Z_{33} & Z_{34} & Z_{35} & Z_{36} & Z_{37} & Z_{38} & Z_{39} & Z_{310} & Z_{311} \\ * & * & * & Z_{44} & Z_{45} & Z_{46} & Z_{47} & Z_{48} & Z_{49} & Z_{410} & Z_{411} \\ * & * & * & * & Z_{55} & Z_{56} & Z_{57} & Z_{58} & Z_{59} & Z_{510} & Z_{511} \\ * & * & * & * & * & Z_{66} & Z_{67} & Z_{68} & Z_{69} & Z_{610} & Z_{611} \\ * & * & * & * & * & * & Z_{77} & Z_{78} & Z_{79} & Z_{710} & Z_{711} \\ * & * & * & * & * & * & * & Z_{88} & Z_{89} & Z_{810} & Z_{811} \\ * & * & * & * & * & * & * & * & Z_{99} & Z_{910} & Z_{911} \\ * & * & * & * & * & * & * & * & * & Z_{1010} & Z_{1011} \\ * & * & * & * & * & * & * & * & * & * & Z_{1111} \end{pmatrix} \begin{pmatrix} \hat{C}x(k) \\ y_L(k+1) \\ y_L(k) \\ y_L(k-1) \\ u_L(k) \\ u_L(k-1) \\ u_R(k+1) \\ u_R(k) \\ u_R(k-1) \\ u'_R(k+1) \\ u'_R(k) \end{pmatrix} \geq 0 \tag{A12}$$

or

$$W_{k-2} + \Omega \geq 0 \tag{A13}$$

alternatively this may be written as

$$\begin{aligned} W_{k-2} + &\begin{pmatrix} \hat{C}x(k) \\ y_L(k+1) \\ y_L(k) \\ y_L(k-1) \\ u_L(k) \\ u_L(k-1) \\ u_R(k+1) \\ u_R(k) \\ u_R(k-1) \\ u'_R(k+1) \\ u'_R(k) \end{pmatrix}^T \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} & Z_{18} & Z_{19} & Z_{110} & Z_{111} \\ * & \mathbf{0} & \mathbf{0} & Z_{24} & Z_{25} & Z_{26} & Z_{27} & Z_{28} & Z_{29} & Z_{210} & Z_{211} \\ * & * & \mathbf{0} & Z_{34} & Z_{35} & Z_{36} & Z_{37} & Z_{38} & Z_{39} & Z_{310} & Z_{311} \\ * & * & * & Z_{44} & Z_{45} & Z_{46} & Z_{47} & Z_{48} & Z_{49} & Z_{410} & Z_{411} \\ * & * & * & * & Z_{55} & Z_{56} & Z_{57} & Z_{58} & Z_{59} & Z_{510} & Z_{511} \\ * & * & * & * & * & Z_{66} & Z_{67} & Z_{68} & Z_{69} & Z_{610} & Z_{611} \\ * & * & * & * & * & * & Z_{77} & Z_{78} & Z_{79} & Z_{710} & Z_{711} \\ * & * & * & * & * & * & * & Z_{88} & Z_{89} & Z_{810} & Z_{811} \\ * & * & * & * & * & * & * & * & Z_{99} & Z_{910} & Z_{911} \\ * & * & * & * & * & * & * & * & * & Z_{1010} & Z_{1011} \\ * & * & * & * & * & * & * & * & * & * & Z_{1111} \end{pmatrix} \begin{pmatrix} \hat{C}x(k) \\ y_L(k+1) \\ y_L(k) \\ y_L(k-1) \\ u_L(k) \\ u_L(k-1) \\ u_R(k+1) \\ u_R(k) \\ u_R(k-1) \\ u'_R(k+1) \\ u'_R(k) \end{pmatrix} \\ &+ \begin{pmatrix} y_L(k+1) \\ y_L(k) \end{pmatrix}^T \begin{pmatrix} Z_{22} & Z_{23} \\ Z_{23}^T & Z_{33} \end{pmatrix} \begin{pmatrix} y_L(k+1) \\ y_L(k) \end{pmatrix} \geq 0 \end{aligned} \tag{A14}$$

where the differences have been bolded, this may be written as

$$W_{k-2} + \Omega + \begin{pmatrix} \tilde{y}_L(k+1) \\ y_L(k) \end{pmatrix}^T \begin{pmatrix} Z_{22} & Z_{23} \\ Z_{23}^T & Z_{33} \end{pmatrix} \begin{pmatrix} \tilde{y}_L(k+1) \\ y_L(k) \end{pmatrix} \geq 0 \tag{A15}$$

where

$$\begin{aligned} Z_{11} &= R_{11}^{LL} & Z_{12} &= S_{11}^{L^T} + R_{11}^{LL} D_1^L \\ Z_{13} &= S_{01}^{L^T} + R_{11}^{LL} D_2^L & Z_{14} &= 0 \\ Z_{15} &= R_{10}^{LL} & Z_{16} &= 0 \\ Z_{17} &= R_{11}^{LR} & Z_{18} &= R_{10}^{LR} \\ Z_{19} &= 0 & Z_{110} &= R_{11}^{LL} D_1^R \\ Z_{111} &= R_{11}^{LL} D_2^R & Z_{22} &= Q_{11} + S_{11}^L D_1^L + D_1^{L^T} R_{11}^{LL} D_1^L \\ Z_{23} &= Q_{10} + S_{11}^L D_2^L + D_1^{L^T} S_{01}^{L^T} + D_1^{L^T} R_{11}^{LL} D_2^L & Z_{24} &= 0 \\ Z_{25} &= S_{10}^L + D_1^{L^T} R_{10}^{LL} & Z_{26} &= 0 \\ Z_{27} &= S_{11}^R + D_1^{L^T} R_{11}^{LR} & Z_{28} &= S_{10}^R + D_1^{L^T} R_{10}^{LR} \\ Z_{29} &= 0 & Z_{210} &= S_{11}^L D_1^R + D_1^{L^T} R_{11}^{LL} D_1^R \\ Z_{211} &= S_{11}^L D_2^R + D_1^{L^T} R_{11}^L D_2^R & Z_{33} &= Q_{00} + Q_{11} + S_{01}^L D_2^L + D_2^{L^T} R_{11}^{LL} D_2^L \\ Z_{34} &= Q_{10} & Z_{35} &= S_{00}^L + S_{11}^L + D_2^{L^T} R_{10}^{LL} \end{aligned}$$

$$Z_{36} = S^L_{10} \qquad Z_{37} = S^R_{01} + D^{L^T}_2 R^{LR}_{11}$$
$$Z_{38} = S^R_{00} + S^R_{11} + D^{L^T}_2 R^{LR}_{10} \qquad Z_{39} = S^R_{10}$$
$$Z_{310} = S^L_{10} D^R_1 + D^{L^T}_2 R^{LL}_{11} D^R_1 \qquad Z_{311} = S^L_{01} D^R_2 + D^{L^T}_2 R^{LL}_{11} D^R_1$$
$$Z_{44} = Q_{00} \qquad Z_{45} = S^L_{01}$$
$$Z_{46} = S^L_{00} \qquad Z_{47} = 0$$
$$Z_{48} = S^R_{01} \qquad Z_{49} = S^R_{00}$$
$$Z_{410} = 0 \qquad Z_{411} = 0$$
$$Z_{55} = R^{LL}_{00} + R^{LL}_{11} \qquad Z_{56} = R^{LL}_{10}$$
$$Z_{57} = R^{LR}_{01} \qquad Z_{58} = R^{LR}_{00} + R^{LR}_{11}$$
$$Z_{59} = R^{LR}_{10} \qquad Z_{510} = R^{LL^T}_{10} D^R_1$$
$$Z_{511} = R^{LL^T}_{10} D^R_2 \qquad Z_{66} = R^{LL}_{00}$$

$$Z_{67} = 0 \qquad Z_{68} = R^{LR}_{01}$$
$$Z_{69} = R^{LR}_{00} \qquad Z_{610} = 0$$
$$Z_{611} = 0 \qquad Z_{77} = R^{RR}_{11}$$
$$Z_{78} = R^{RR}_{10} \qquad Z_{79} = 0$$
$$Z_{710} = R^{LR^T}_{11} D^R_1 \qquad Z_{711} = R^{LR^T}_{11} D^R_2$$
$$Z_{88} = R^{RR}_{00} + R^{RR}_{11} \qquad Z_{89} = R^{RR}_{10}$$
$$Z_{810} = R^{LR^T}_{10} D^R_1 \qquad Z_{811} = R^{LR^T}_{10} D^R_2$$
$$Z_{99} = R^{RR}_{00} \qquad Z_{910} = 0$$
$$Z_{911} = 0 \qquad Z_{1010} = D^{R^T}_1 R^{LL}_{11} D^R_1$$
$$Z_{1011} = D^{R^T}_1 R^{LL}_{11} D^R_2 \qquad Z_{1111} = D^{R^T}_2 R^{LL}_{11} D^R_2$$

Letting

$$\chi = \begin{pmatrix} Z_{22} & Z_{23} \\ Z^T_{23} & Z_{33} \end{pmatrix} = \begin{pmatrix} Q_{11} + S^L_{11} D^L_1 + D^{L^T}_1 R^{LL}_{11} D^L_1 & Q_{10} + S^L_{11} D^L_2 + D^{L^T}_1 S^{L^T}_{01} + D^{L^T}_1 R^{LL}_{11} D^L_2 \\ \left(Q_{10} + S^L_{11} D^L_2 + D^{L^T}_1 S^{L^T}_{01} + D^{L^T}_1 R^{LL}_{11} D^L_2\right)^T & Q_{00} + Q_{11} + S^L_{01} D^L_2 + D^{L^T}_2 R^{LL}_{11} D^L_2 \end{pmatrix} < 0 \tag{A16}$$

and taking a Schur complement in that block in (100), the stated condition is achieved. ■

Proof of Proposition 3. Assume that the dissipativity of the MPC is formulated such that $-\chi > 0$ is satisfied (note that this may be determined offline). Then the dissipativity trajectory condition at time $k = 1$ reduces to

$$\begin{pmatrix} -\chi^{-1} & \hat{y}^T_L(1) \\ \hat{y}_L(1) & \Omega + W_0 \end{pmatrix} \geq 0 \tag{A17}$$

As $W_0 \geq 0$, to ensure feasibility take the worst case that $W_0 = 0$. The dissipative trajectory condition then requires $\begin{pmatrix} -\chi^{-1} & \hat{y}^T_L(1) \\ \hat{y}_L(1) & \Omega \end{pmatrix} \geq 0$. As the origin is inside the feasible region it is always possible to chose $\hat{y}_L(1) = 0$. The condition then becomes

$$\begin{pmatrix} -\chi^{-1} & 0 \\ 0 & \Omega_0 \end{pmatrix} \geq 0 \tag{A18}$$

where Ω_0 is Ω with $\hat{y}_L(k) = 0$. Thus, the stated condition is sufficient to ensure feasibility $\forall x(1) \in \mathcal{X}$. Having shown that the condition is feasible at time $k = 1$, it is then possible to show it is feasible for $k = 2$ and so on iteratively $\forall k$ by the same argument. ■

Proof of Theorem 2. From Lemma 1 the plant-wide system (consisting of process and controller networks) satisfies the dissipativity inequality

$$Q_\mu(t) \geq Q_{\nabla\Psi}(t) \tag{A19}$$

For all $t \geq 0$ Defining $\Xi = \begin{pmatrix} \Upsilon_{11}(\zeta,\eta) & \Upsilon_{13}(\zeta,\eta) \\ \Upsilon^T_{13}(\zeta,\eta) & \Upsilon_{33}(\zeta,\eta) \end{pmatrix}$ and for vanishing disturbance this becomes

$$Q_\Xi(t) \geq Q_{\nabla\Psi}(t) \tag{A20}$$

Using the condition that $\begin{pmatrix} \Upsilon_{11}(\zeta,\eta) & \Upsilon_{13}(\zeta,\eta) \\ \Upsilon^T_{13}(\zeta,\eta) & \Upsilon_{33}(\zeta,\eta) \end{pmatrix} < 0$ and the fact that $\Xi(\zeta, \eta) < 0 \iff \partial\Xi(-\xi, \xi) = -D^T(-\xi)D(\xi)$ with $D(\lambda)$ of full rank $\forall\ \lambda \in \mathbb{C}$. For vanishing disturbance this becomes

$$-\left(D(\xi)\begin{matrix} \mathbf{y_{pw}}(t) \\ x(t) \end{matrix}\right)^2 \geq Q_{\nabla\Psi}(t) \tag{A21}$$

As $D(\xi)$ has full rank this implies $Q_{\nabla\Psi}(t) < 0\ \forall t$, which, combined with the assumption that $\Psi \geq 0$ implies asymptotic stability by Theorem 1. ■

Proof of Theorem 3. Using the superscript ˆ to denote extensions of the variables to include future values up to the order of the supply rates as in Proposition 1, the plant-wide system traces a dissipative trajectory implying

$$\sum_{i=0}^{k} \hat{\mathbf{y}}^T_{\mathbf{pw}}(k)\Gamma_{11}\hat{\mathbf{y}}_{\mathbf{pw}}(k) + 2\hat{\mathbf{y}}^T_{\mathbf{pw}}(k)\Gamma_{12}\hat{\mathbf{d}}(k) + \hat{\mathbf{d}}^T(k)\Gamma_{22}\hat{\mathbf{d}}(k) \geq Q_\Psi(k+1) - Q_\Psi(0) \tag{A22}$$

is satisfied for any disturbance. Assuming the system has the condition $Q_\Psi(k + 1) = Q_\Psi(0) = 0$, we have

$$\sum_{i=0}^{k} \hat{\mathbf{y}}^T_{\mathbf{pw}}(k)\Gamma_{11}\hat{\mathbf{y}}_{\mathbf{pw}}(k) + 2\hat{\mathbf{y}}^T_{\mathbf{pw}}(k)\Gamma_{12}\hat{\mathbf{d}}(k) + \hat{\mathbf{d}}^T(k)\Gamma_{22}\hat{\mathbf{d}}(k) \geq 0 \tag{A23}$$

Defining $\hat{\Gamma}_{11} = -\Gamma_{11}$, we have

$$\sum_{i=0}^{k} \hat{\mathbf{y}}^T_{\mathbf{pw}}(k)\hat{\Gamma}_{11}\hat{\mathbf{y}}_{\mathbf{pw}}(k) - 2\hat{\mathbf{y}}^T_{\mathbf{pw}}(k)\Gamma_{12}\hat{\mathbf{d}}(k) \leq \hat{\mathbf{d}}^T(k)\Gamma_{22}\hat{\mathbf{d}}(k) \tag{A24}$$

Completing the square on the left hand side leads to

$$\sum_{i=0}^{k} [\hat{\Gamma}^{\frac{1}{2}}_{11}\hat{\mathbf{y}}_{\mathbf{pw}} - \hat{\Gamma}^{-\frac{1}{2}}_{11}\Gamma_{12}\hat{\mathbf{d}}]^T[\hat{\Gamma}^{\frac{1}{2}}_{11}\hat{\mathbf{y}}_{\mathbf{pw}} - \hat{\Gamma}^{-\frac{1}{2}}_{11}\Gamma_{12}\hat{\mathbf{d}}] \leq \sum_{i=0}^{k} \hat{\mathbf{d}}^T[\Gamma_{22} + \Gamma^T_{12}\hat{\Gamma}_{11}\Gamma_{12}]\hat{\mathbf{d}} \tag{A25}$$

Defining a row vector p of length $\deg(\mu(\zeta, \eta))$ such that $p^T p \geq \max(\Gamma_{22} + \Gamma_{12}^T \hat{\Gamma}_{11} \Gamma_{12}, \Gamma_{12}^T \hat{\Gamma}_{11} \Gamma_{12})$ and using the reverse triangle inequality

$$\|\hat{\Gamma}_{11}^{\frac{1}{2}}\hat{\mathbf{y}}_{\mathbf{pw}}\| - \|\hat{\Gamma}_{11}^{-\frac{1}{2}}\Gamma_{12}\hat{\mathbf{d}}\| \leq \|p\hat{\mathbf{d}}\| \tag{A26}$$

$$\|\hat{\Gamma}_{11}^{\frac{1}{2}}\hat{\mathbf{y}}_{\mathbf{pw}}\| \leq \|\hat{\Gamma}_{11}^{-\frac{1}{2}}\Gamma_{12}\hat{\mathbf{d}}\| + \|p\hat{\mathbf{d}}\| \tag{A27}$$

$$\|\hat{\Gamma}_{11}^{\frac{1}{2}}\hat{\mathbf{y}}_{\mathbf{pw}}\| \leq 2\|p\hat{\mathbf{d}}\| \tag{A28}$$

Returning to the original variables

$$\|\hat{\Gamma}_{11}^{\frac{1}{2}}(\eta)\mathbf{y}_{\mathbf{pw}}\| \leq 2\|p(\eta)\mathbf{d}\| \tag{A29}$$

$$\left\|\frac{\hat{\Gamma}_{11}^{\frac{1}{2}}(\eta)}{p(\eta)}\mathbf{y}_{\mathbf{pw}}\right\| \leq 2\|\mathbf{d}\| \tag{A30}$$

where in the last line the fact that $p(\eta)$ is a scalar is used. It is clear that if $\frac{\hat{\Gamma}_{11}^{\frac{1}{2}}(\eta)}{p(\eta)} \geq \frac{1}{\gamma}$ then

$$\|\mathbf{y}_{\mathbf{pw}}\| \leq \gamma\|\mathbf{d}\| \tag{A31}$$

Proof for the Optimization Step (Step 2) of Problem 2. Constraint (64) is obvious. Constraints (62) and (63) imply the controller traces a dissipative trajectory as described in Proposition 2. The output constraint is transformed into an LMI by rearranging as $1 + \epsilon - \hat{y}^T(k)P\hat{y}(k) \geq 0$ and taking a Schur complement in P (using $P > 0$). The cost function is

$$J(k) = Q_\psi(k) + w_y\epsilon \tag{A32}$$

$$= \sum_{k=0}^{k_{\max}}\sum_{l=0}^{l_{\max}} \begin{pmatrix} y(t+k) \\ y_L(t+k) \\ d(t+k) \\ u_r(t+k) \end{pmatrix}^T \begin{pmatrix} \psi_{kl}^{11} & \psi_{kl}^{12} & \psi_{kl}^{13} & \psi_{kl}^{14} \\ \psi_{kl}^{12^T} & \psi_{kl}^{22} & \psi_{kl}^{23} & \psi_{kl}^{24} \\ \psi_{kl}^{13^T} & \psi_{kl}^{23^T} & \psi_{kl}^{33} & \psi_{kl}^{34} \\ \psi_{kl}^{14^T} & \psi_{kl}^{24^T} & \psi_{kl}^{34^T} & \psi_{kl}^{44} \end{pmatrix} \times \begin{pmatrix} y(t+k) \\ y_L(t+k) \\ d(t+k) \\ u_r(t+k) \end{pmatrix} + w_y\epsilon \tag{A33}$$

For some $w_y > 0$ where the substitutions $u_c = y_L$ and $u_p = u_r$ have been made. As the disturbance is assumed to be unmeasured (but vanishing) ignore the terms associated with it

$$J(k) = \sum_{k=0}^{k_{\max}}\sum_{l=0}^{l_{\max}} \begin{pmatrix} y(t+k) \\ y_L(t+k) \\ u_r(t+k) \end{pmatrix}^T \begin{pmatrix} \psi_{kl}^{11} & \psi_{kl}^{12} & \psi_{kl}^{14} \\ \psi_{kl}^{12^T} & \psi_{kl}^{22} & \psi_{kl}^{24} \\ \psi_{kl}^{14^T} & \psi_{kl}^{24^T} & \psi_{kl}^{44} \end{pmatrix} \times \begin{pmatrix} y(t+l) \\ y_L(t+l) \\ u_r(t+k) \end{pmatrix} + w_y\epsilon \tag{A34}$$

Permuting the coefficient matrix of $\psi(\zeta, \eta)$ conformally with $\hat{y_p}(k)$ and $\hat{y_c}(k)$

$$J(k) = \begin{pmatrix} \hat{y}(k) \\ \hat{y}_L(k) \\ \hat{u}_r(k) \end{pmatrix}^T \begin{pmatrix} \tilde{\psi}_{11} & \tilde{\psi}_{12} & \tilde{\psi}_{14} \\ \tilde{\psi}_{12}^T & \tilde{\psi}_{22} & \tilde{\psi}_{24} \\ \tilde{\psi}_{14^T} & \tilde{\psi}_{24^T} & \tilde{\psi}_{44} \end{pmatrix} \begin{pmatrix} \hat{y}(k) \\ \hat{y}_L(k) \\ \hat{u}_r(k) \end{pmatrix} + w_y\epsilon \tag{A35}$$

Allowing α be a scalar such that

$$\alpha - J(k) \geq 0 \tag{A36}$$

leads to

$$\begin{aligned} &\alpha - \hat{y}^T(k)\tilde{\psi}_{11}\hat{y}(k) - 2\hat{y}^T(k)\tilde{\psi}_{12}\hat{y}_L(k) - \hat{y}^T\tilde{\psi}_{14}\hat{u}_r - 2\hat{y}_L^T(k)\tilde{\psi}_{22}\hat{y}_L(k) \\ &- \hat{y}_L^T(k)\tilde{\psi}_{24}\hat{u}_r(k) - \hat{u}_r^T(k)\tilde{\psi}_{44}\hat{u}_r(k) + w_y\epsilon \geq 0 \end{aligned} \tag{A37}$$

Substituting in $\hat{y} = \hat{C}x(k) + \hat{D_1}\hat{y_L} + \hat{D_2}\hat{u_r}$, we have

$$\begin{aligned} &\alpha - \hat{y}_L^T\left(\hat{D}_1^T\psi_{11}\hat{D}_1 + \hat{D}_1^T\psi_{12} + \psi_{12}^T\hat{D}_1 + \psi_{22}\right)\hat{y}_L \\ &- x^T(k)\hat{C}^T\psi_{11}\hat{C}x(k) - 2x^T(k)\hat{C}^T\left(\psi_{11}\hat{D}_1 + \psi_{12}\right)\hat{y}_L \\ &- 2x^T(k)\hat{C}^T\left(\psi_{11}\hat{D}_2 + \psi_{14}\right)\hat{u}_r \\ &- 2\hat{y}_L^T\left(\hat{D}_1^T\psi_{11}\hat{D}_2 + \psi_{24} + \psi_{12}^T\hat{D}_2^T + \hat{D}_1^T\psi_{14}\right)\hat{u}_r \\ &- \hat{u}_r^T\left(\hat{D}_2^T\psi_{11}\hat{D}_2 + \hat{D}_2^T\psi_{14} + \psi_{14}^T\hat{D}_2 + \psi_{44}\right)\hat{u}_r + w_y\epsilon \geq 0 \end{aligned} \tag{A38}$$

Assuming $(\hat{D_1^T}\psi_{11}\hat{D_1} + \hat{D_1^T}\psi_{12} + \psi_{12}^T\hat{D_1} + \psi_{22}) > 0$ and taking a Schur complement, the stated LMI condition is obtained.

Appendix B

Dissipativity properties for illustrative example

The supply rate of the first controller (which controls the first reactor) is provided below as an indicative example of the form of the supply rates for the other controllers and processes. The supply rates are found by solving the LMI Problem 1 offline to find the optimal supply rates of each process and controller. In the form of (5) it is induced by the matrix

$$\begin{pmatrix} \tilde{Q}_c & \tilde{S}_c \\ \tilde{S}_c^T & \tilde{R}_c \end{pmatrix} \tag{B1}$$

where

$$\tilde{Q}_c = \begin{pmatrix}
-0.03854 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.03854 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.03854 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.04121 & -0.00055 & -0.00004 & 0.01146 & -0.00053 & -0.00004 & 0.00541 & -0.00053 & -0.00004 \\
0 & 0 & 0 & -0.00055 & -0.02651 & 0.00026 & -0.00007 & -0.02701 & 0.00026 & 0.00048 & -0.02708 & 0.00026 \\
0 & 0 & 0 & -0.00004 & 0.00026 & -0.02116 & -0.00003 & 0.00169 & -0.03007 & 0.00003 & 0.00256 & -0.03007 \\
0 & 0 & 0 & 0.01146 & -0.00007 & -0.00003 & -0.02399 & 0 & 0 & 0.01244 & 0 & 0.00002 \\
0 & 0 & 0 & -0.00053 & -0.02701 & 0.00169 & 0 & -0.03134 & -0.00256 & 0.00053 & -0.00417 & -0.00707 \\
0 & 0 & 0 & -0.00004 & 0.00026 & -0.03007 & 0 & -0.00256 & -0.03821 & 0.00005 & -0.00471 & -0.02229 \\
0 & 0 & 0 & 0.00541 & 0.00048 & 0.00003 & 0.01244 & 0.00053 & 0.00005 & -0.01784 & 0.000526 & 0.00004 \\
0 & 0 & 0 & -0.00053 & -0.02708 & 0.00256 & 0 & -0.00417 & -0.00471 & 0.000526 & -0.03128 & -0.00845 \\
0 & 0 & 0 & -0.00004 & 0.00026 & -0.03007 & 0.00002 & -0.00707 & -0.02229 & 0.00004 & -0.00845 & -0.03986
\end{pmatrix}$$

$$\tilde{S}_c = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.00042 & 0.00033 & 0.00001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.000021 & 0.00002 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.00018 & -0.00014 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.00121 & -0.00091 & -0.00003 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0001 & 0.00008 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\tilde{R}_c = \begin{pmatrix}
-0.04137 & -0.00006 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.00006 & -0.04207 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.0421 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.03487 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.03487 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.03487 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.03487 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.03487 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.03487 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.03487 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.03487 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.03487
\end{pmatrix}$$

Manuscript received Jan. 17, 2012, revision received May 14, 2012, and final revision received Jun. 6, 2012.